

# Quantum entropy of supersymmetric black holes\*

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**ABSTRACT:** We review recent progress concerning the quantum entropy of a large class of supersymmetric black holes in string theory both from the microscopic and macroscopic sides. On the microscopic field theory side, we present new results concerning the counting of black hole microstates for charge vectors with nontrivial arithmetic duality invariants. On the macroscopic gravitational side, we present a novel application of localization techniques to a supergravity functional integral to compute the quantum entropy of these black holes. Localization leads to an enormous simplification of a path integral of string theory in  $AdS_2$  by reducing it to a finite dimensional integral. The localizing solutions are labeled by  $n_v + 1$  parameters, with  $n_v$  the number of vector multiplets in the theory of  $\mathcal{N} = 2$  supergravity. As an example we show, for four dimensional large black holes which preserve four supersymmetries in toroidally compactified IIB string theory, that the macroscopic degeneracy precisely agrees with all the terms in an exact Rademacher expansion of the microscopic answer except for Kloosterman sums which in principle can be computed. Generalizing previous work, these finite charge contributions to the leading Bekenstein-Hawking entropy can also be viewed as an instance of “exact holography” in the context of  $AdS_2/CFT_1$  correspondence.

**KEYWORDS:** black holes, superstrings, dyons, holography.

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\*This article is based on the thesis of the author.

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## 1. Introduction

Einstein's general theory of relativity predicts that a sufficiently massive object can deform spacetime in such a way that it creates a region from where not even light can escape. This solution is called a black hole. The boundary of such a region is a null hypersurface called event horizon. This a surface of infinite redshift which then motivates the name black hole.

As pointed out first by Bekenstein [1] and then by Hawking [2] a black hole must carry entropy so that the second law of thermodynamics is not violated. The classic thought experiment is to throw a bucket of warm water inside the horizon. Since the entropy of the Universe cannot decrease the black hole must have entropy. It is well known that, in general theory of

relativity, the black hole entropy is proportional to the area of the horizon in contrast with ordinary matter systems where it is proportional to the accessible volume,

$$S_{BH} = \frac{A}{4G_N\hbar}. \quad (1.1)$$

Here  $A$  is the area of the horizon and  $G_N$  is the Newton's constant. Consistency with statistical mechanics naturally lead us to the following question: can we describe a black hole as an ensemble of quantum states in such way that we can relate the entropy  $S_{BH}$  to the logarithm of the number of accessible states?

$$S_{BH} = \ln \Omega_{micro} \quad (1.2)$$

To answer this question we need a theory of quantum gravity. String theory is the leading candidate for such a theory. Although we are still far from a description of the real world in terms of strings, this theory is able to incorporate gravity in a consistent way with other forces and it leads to the discovery of branes from where the holographic correspondence [3] was born. String theory gives us a systematic procedure to compute corrections to Einstein's theory of gravity which can be important to understand finite size effects in quantum gravity.

The salient results covered by this article are:

- ***Finite charge corrections to Bekenstein-Hawking entropy***

The main focus is the computation of finite charge corrections to the leading Bekenstein-Hawking entropy. Formula (1.1) is valid for an action with only the Einstein-Hilbert term. Since in string theory both the  $\alpha'$  and string-loop corrections depend on the phase<sup>1</sup> of the theory, finite size corrections to the area law can give us information about the microscopic details of the phase.

To implement the effect of the higher derivative corrections we need to use the Wald formalism [4, 5]. The entropy is then given by a surface integral over the horizon geometry. To compute the Wald entropy we need first to find the black hole solution by solving the gravity equations and then perform the surface integral which is not an easy task. However, for extremal black holes, the near horizon geometry has enhanced symmetries which can be used to simplify the computation of the Wald entropy. The near horizon geometry  $AdS_2 \times S^2$  has  $SO(2, 1) \times SO(3)$  symmetries and is separated from infinity by an infinite throat. The moduli of the theory get attracted at the horizon and their value only depends on the charges. This is called attractor mechanism [6]. Combining both the symmetries of the near horizon geometry and the attractor mechanism Sen gives a simple prescription to compute the Wald entropy [7]. This method, called entropy function, resumes all the computation of Wald to a minimization problem and does not require solving the Einstein's equations. The entropy function is proportional to the Lagrangian computed on the attractor background and the minimization parameters are the attractor

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<sup>1</sup>By phase we mean the compactification of string theory.

values of the different fields. The black hole entropy is then given by the minimum of that function.

Unfortunately this formalism is not completely adequate in the full quantum theory. There can be non-local and/or non-analytic terms in the action coming from the integration of massless fields. In this case the Wald formalism can not be applied since it requires a local and gauge invariant action. Moreover it is in our interest to compute not just perturbative but also non-perturbative corrections to the Wald entropy as suggested by the microscopic answers. Thus even defining the proper notion of quantum entropy presents important conceptual problems. In an attempt to solve these issues Sen proposes a different formalism called quantum entropy [8, 9]. The idea is based on  $AdS_2/CFT_1$  correspondence and gives a quantum version of the entropy. In summary, we are instructed to compute a path integral of string theory on  $AdS_2$  with a Wilson line insertion at the boundary. The holographic correspondence then relates this observable to the degeneracy of the black hole. In contrast with higher dimensional cases, in  $AdS_2$  the electric fields are non-normalizable modes and therefore they have to be fixed while performing the path integral. This is equivalent to fixing the charges instead of the chemical potentials which means that we are working in the microcanonical ensemble. For large charges the path integral is peaked at the classical attractor saddle point and the computation reduces to that of the entropy function. Since the equations of motion are no longer implied we can compute both perturbative and non-perturbative contributions in a systematic way.

- ***Supersymmetric Localization***

For supersymmetric theories we can hope to use supersymmetric localization to compute exactly a path integral [10, 11, 12]. In a few words, localization means deforming the original physical action by a Q-exact term of the form  $tQV$ , where  $Q$  is the action of some supersymmetry. If both the deformation and the observable, we are interested in computing, are supersymmetric by themselves, then it can be shown that the path integral does not depend on  $t$ . This is very practical because we can choose a parameter  $t$  where the computation is more convenient. In the limit  $t \rightarrow \infty$  the deformation  $tQV$  dominates over the original physical action and the semiclassical approximation becomes exact since, in this case,  $t$  plays the role of  $\hbar^{-1}$ . Application of this technique in a path integral requires the supersymmetry  $Q$  to be realized off-shell. Fortunately for us, there is an off-shell formulation of supergravity even though only eight SUSYs are realized [13, 14, 15, 16]. When this formalism is applied to supergravity on  $AdS_2 \times S^2$  the path integral localizes to a subspace where the scalar fields can be excited above their attractor values at the cost of exciting the auxiliary fields [17]. The solution is labeled by  $n_V + 1$  constants where  $n_V$  is the number of vector multiplets in the theory. Using this technique we were able to reduce a very complicated path integral to a finite dimensional integral which resembles very much the formula proposed by Ooguri, Strominger and Vafa [18] but with some important differences. These differences include for example loop determinants,

instantons or subleading orbifold saddle points. Once they are taken into account it should be possible to reproduce exactly the microscopic answers.

For large black holes in  $\mathcal{N} = 8$  string theory the microscopic answer has a simple expression given in terms of a Jacobi form. The degeneracy can be written as a sum of Bessel functions in an exact expansion called Rademacher expansion. Using localization techniques we were able to reproduce *all* these terms for arbitrary values of the charges except for Kloosterman sums that can in principle be computed [19]. In this analysis we do a careful treatment of the measure on the localization locus which reveals crucial for the exact matching.

The big goal is to establish an exact equality between a degeneracy computed from the microscopic degrees of freedom and the quantum entropy computed from gravity. This obviously implies two big tasks,

- The first is to compute the expectation value of a Wilson line in  $AdS_2$  by performing a path integral over the horizon string fields. The localization technique is extremely useful in this case.
- The second is to compute precisely the microscopic degeneracy using some weak coupling description in the same spirit of Strominger and Vafa [20].

For large charges both tasks simplify. In this regime we can use a Cardy formula to compute the microscopic degeneracy. On the gravity side large charge means large horizon radius and therefore we can neglect higher derivative corrections. The entropy area law suffices in this case. Performing both tasks exactly is equivalent as establishing an exact  $AdS_2/CFT_1$  holography.

### • *Microscopic counting*

The success of Strominger and Vafa black hole inspired many other works. Results in microscopic BPS counting flourished. For quarter BPS black holes in  $\mathcal{N} = 4$  string theory the results are particularly interesting. The microscopic partition function is given in terms of the Fourier coefficients of a Siegel modular form, which is a very rich object from the mathematical point of view. Part of this review is devoted to the analysis of the quarter-BPS dyon spectrum in these theories and to the construction of the corresponding partition functions. Previous works [21, 22, 23, 24, 25, 26, 27, 28, 29, 30] concern the spectrum of dyons which obey a particular primitivity constrain on the charges. As first noted in [31],  $I = \gcd(\mathbf{q} \wedge \mathbf{p})$  is the only discrete invariant relevant in this problem, where  $(\mathbf{q}, \mathbf{p})$  denotes the dyonic electric and magnetic charges vectors respectively. Consider the charge lattice  $\Lambda$  where both the electric  $\mathbf{q}$  and magnetic  $\mathbf{p}$  charge vectors live. These charge vectors generate a two dimensional lattice inside  $\Lambda$ . The invariant  $I$  basically counts the number of unit cells of  $\Lambda$  inside a cell bounded by  $\mathbf{q}$  and  $\mathbf{p}$ . A primitive dyon corresponds to a unit cell. When the primitivity condition is relaxed additional difficulties arise in the microscopic counting mainly due to the analysis of multi-particle bound states at threshold

[32]. Without loss of generality we consider the case when the electric charge vector  $\mathbf{q}$  is a multiple  $I$  of a primitive vector while  $\mathbf{p}$  is primitive. In type IIB frame this implies studying a system of D-branes weakly interacting with a  $I$  KK-monopoles. We study the low energy theory and propose a two dimensional supersymmetric sigma model [30]. A modified elliptic genus then gives an index which is consistent with previous constructions [33, 29] and passes many physical tests. In brief, the index found is given in terms of the fourier coefficients of the primitive answer and carries a non-trivial dependence on the divisors of  $I$ . In [34] we propose a non-trivial check of the counting formula. We map a particular set of states to perturbative momentum-winding states of IIA string theory where the counting can be easily done and agreement is found for any value of  $I$ .

- ***Cardy limit and  $AdS_3/CFT_2$  correspondence***

In the last section we focus on a different approach based mostly on  $AdS_3$  rather than  $AdS_2$ . Instead of computing the entropy valid for any charge we consider the simpler case when only one of the charges is very large keeping the other charges arbitrarily finite. The result is exact in the limit considered and is able to probe details of the phase we are working on [35]. The main result is: *for black holes which preserve at least four supercharges the asymptotic growth of the index has a Cardy like formula with an effective central charge that is given by a linear combination of the coefficients of the Chern-Simons terms computed at asymptotic infinity.* Whenever a black hole has a factor  $AdS_2 \times S^1$  in the near horizon geometry we can view it as an extremal BTZ black hole living in  $AdS_3$  space in the limit when the circle  $S^1$  has a very large radius. The momentum along the circle corresponds to the angular momentum  $J$  of the BTZ black hole. Then the extremal condition  $M = J$  implies that we are counting states of large mass and therefore we can use a Cardy formula. In this case holography is extremely powerful since it relates the central charges, which are anomaly coefficients in the CFT, to the coefficients of the Chern-Simons terms living in the bulk of  $AdS_3$  [36, 37, 38]. Note that the entropy formulas obtained are exact in the limit considered, that is, when only one of the charges is taken to be very large while keeping the remaining finite. During the analysis we found convenient to consider a macroscopic index that captures all the degrees of freedom from the horizon till asymptotic infinity. In the process we need to take into account additional contributions from external modes to the bulk of  $AdS_3$ .

The review is organized as follows. In section §2 we give exact results on the microscopic counting of both primitive and non-primitive dyons. In section §3 we explain the quantum entropy formalism based on the  $AdS_2/CFT_1$  correspondence and its relation with the microscopic index. In section §4 we use localization of supergravity on  $AdS_2 \times S^2$  to reduce a very complicated path integral to a finite dimensional integral. We end discussing its relation to the OSV proposal. In section §5 we apply our results from localization in the problem of large black holes in  $\mathcal{N} = 8$  string theory. Since the microscopic answer is known exactly we conclude by comparing both the macroscopic and microscopic answers which agree exactly for any finite

charge. In the last section we study the index in the particular charge limit where only one of the charges is taken to be very large. In this regime of charges the  $AdS_3$  point of view becomes more useful.

## 2. Microscopic counting

In string theory the Newton's constant  $G_N$  is proportional to the square of string coupling  $g_s$ . As a consequence the gravitational attraction, proportional to  $G_N M$ , with  $M$  the mass of the object, can be made arbitrarily small by decreasing  $g_s$ . In particular, for fundamental strings and D-branes  $G_N M$  goes as  $g_s^2$  and  $g_s$  respectively while for the KK monopole or NS5-brane they are of order one. In this regime of very weak string coupling we can turn off gravity and “dissolve” the black hole. The space becomes flat and these objects weakly interact. In this regime we can count the microscopic BPS states by quantizing the low energy theory of the system.

The first successful example in matching the microscopic degeneracy with the Bekenstein-Hawking entropy is the Vafa and Strominger five dimensional black hole [20]. They consider a system of D1 and D5-branes wrapping cycles of  $K3 \times S^1$  in type IIB string theory along with momenta through the circle. Effectively we see a five dimensional black hole carrying electric and magnetic charges. The low energy theory of the branes is a two dimensional supersymmetric conformal field theory. In the limit of large charges we can use a Cardy formula to compute the entropy of BPS states while on the black hole regime the entropy has the area law. Both answers perfectly agree.

For a large class of supersymmetric black holes it is known that the number of BPS states is constant over regions of the moduli space separated by codimension one walls where the states are marginally stable against decay [39, 40, 41, 42, 43, 44]. The constancy of the degeneracy follows from the non-renormalization of the mass of a state that saturates the Bogomol'nyi-Prasad-Sommerfeld bound, that is, of a BPS state. In other words the mass  $M$  equals the central charge  $Z(Q)$  which is perturbatively not renormalized and therefore these BPS states sit in multiplets of shorter dimension. Due to this property, we can work in a region of the moduli space where string theory is weakly coupled, count the number of BPS states and then extrapolate this result to strong coupling, in the black hole regime. In the limit of large charges, or thermodynamic limit, the curvature of the horizon becomes small and the entropy is given by the Bekenstein-Hawking area law.

To count BPS states we use an index. This has the property of being invariant under continuous deformations of the theory. This is exactly what we mean by the constancy of the number of BPS states over the moduli space. In particular we use a helicity trace index or spacetime index. As a matter of fact, the index counts the number of bosonic minus fermionic states and therefore it can be zero or even negative. This is puzzling because ultimately we want to compare it with the exponential of the Bekenstein-Hawking entropy which is a strictly positive quantity. The usual understanding is that the number of states that get paired up is subleading in the large charge limit. Later we will see that the correct thing to do is to compare



this microscopic index with an index constructed from the black hole solution. This issue will be analysed in section §6 where we make a clear distinction between index and degeneracy.

Since the index is invariant under U-duality it becomes important to classify duality orbits and corresponding charge invariants. For dyons in  $\mathcal{N} = 4$  string theory with electric and magnetic charge vectors  $Q$  and  $P$  we can construct many duality invariants out of the charges. Apart from the continuous T-duality invariants  $Q^2$ ,  $P^2$  and  $Q.P$  there is one discrete U-duality invariant  $I = \gcd(Q \wedge P)$  which is particularly important in this problem. Very basically it encodes a primitivity condition in the dyon charge vector. A primitive dyon is one for which  $I = 1$ . Previous works in  $\mathcal{N} = 4$  string theory concern the spectrum of primitive dyons [45, 46, 23, 22, 24, 26, 47, 48]. The main focus of this section is the counting of quarter-BPS states when the primitivity condition is relaxed. We propose a two dimensional supersymmetric sigma model whose index captures the spectrum of non-primitive dyons [30]. The resulting index is consistent with many physical tests including a perturbative test [34] and is in agreement with the answer proposed in [33, 29].

This section is organized as follows. In section §2.1 we consider the low energy theory of Heterotic string on  $T^6$  and give general properties of quarter BPS dyons. In section §2.2 we focus on U-duality and classification of orbits via charge invariants. In particular we identify an important U-duality invariant  $I$  on which the counting depends non trivially. Further in section §2.3 we analyse the role of invariant  $I$  in the microscopic counting, explaining the construction of the partition functions in the cases  $I = 1$  and  $I > 1$ .

## 2.1 Heterotic string on $T^6$ : generalities

We consider heterotic string theory compactified on a six-dimensional torus  $T^6$ . This is a four-dimensional string theory with  $\mathcal{N} = 4$  supersymmetry or sixteen supercharges. It can have a dual description as IIA or IIB string theory compactified on  $K3 \times T^2$ .

The four-dimensional low energy theory contains the metric  $g_{\mu\nu}$ , the axion-dilaton  $\lambda = a + ie^{-2\phi}$  and six  $U(1)$  gauge fields  $A_\mu$  together with their susy partners sitting in the gravity multiplet. It contains in addition 22 vector multiplets. Each of these contains a  $U(1)$  gauge field and six real scalars plus susy partners. The axion-dilaton together with the  $132=22 \times 6$  scalars from the vectors parametrize the moduli space of the theory

$$\frac{SL(2, \mathbb{R})}{SL(2, \mathbb{Z}) \times SO(2)} \times \frac{SO(22, 6; \mathbb{R})}{SO(22, 6; \mathbb{Z}) \times SO(22, \mathbb{R}) \times SO(6, \mathbb{R})}. \quad (2.1)$$

This theory has U-duality group

$$G(\mathbb{Z}) = SL(2, \mathbb{Z}) \times O(22, 6; \mathbb{Z}) \quad (2.2)$$

where the first factor corresponds to electric-magnetic duality and the second factor corresponds to T-duality.

The 28 gauge fields can carry electric and magnetic  $Q, P$  charges which can be arranged in the dyon charge vector

$$\Gamma^i = \begin{bmatrix} Q^i \\ P^i \end{bmatrix}$$

The index 'i' stands for the vector representation of  $SO(22, 6; \mathbb{Z})$  and the electric-magnetic duality acts on the pair  $(Q, P)$  by an  $SL(2, \mathbb{Z})$  transformation. This is also the S-duality symmetry of the four dimensional theory that acts on the axion-dilaton. Both the dyon and the axion-dilaton transform as

$$\lambda \rightarrow \frac{a\lambda + b}{c\lambda + d}, \quad \begin{bmatrix} Q \\ P \end{bmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

The  $\mathcal{N} = 4$  superalgebra has central charges  $Z_1(\Gamma, \phi_\infty) > Z_2(\Gamma, \phi_\infty)$ . A dyon with mass  $M$  that saturates the BPS bound  $M = Z_1(\Gamma, \phi_\infty)$  will preserve 1/4 of the supersymmetries. Note the dependence on the moduli  $\phi_\infty$  measured at infinity. For certain values of  $\phi_\infty$  the state can become marginally stable against decay into 1/2-BPS states. These regions are codimension one and are called walls of marginal stability [40]. As a consequence the index will jump.

A 1/4-BPS dyon breaks 12 supercharges out of 16. The 12 fermion zero modes associated with the broken susys make the Witten index  $\text{Tr}(-1)^F$  vanish. To correctly account for the additional fermion zero modes we need to use a modified index [27]. Also known as helicity trace index or spacetime index, it is defined as

$$B_6(\Gamma, \phi_\infty) = -\frac{1}{6!} \text{Tr}(-1)^F (2h)^6 \quad (2.3)$$

where  $h$  is the helicity quantum number and  $F = 2h$  is the fermion number. The insertion of  $(2h)^6$  in the usual Witten index has the effect of rendering the trace over the fermion zero modes non-zero.

Lets work in more detail the contribution of the fermion zero modes. Each pair carries  $h = \pm 1/4$ . To simplify the counting we compute first  $g(y) = -\frac{1}{6!} \text{Tr}(-1)^F y^{2h}$  and the index  $B_6$  becomes the sixth derivative of  $g(y)$  at  $y = 1$ . Tracing over the six complex fermion zero modes we obtain  $g(y) = \frac{1}{6!} (y^{1/2} - y^{-1/2})^6$  which, after differentiation, gives the net result of 1. In most of the cases we use the Witten index  $\text{Tr}'(-1)^F$  where the ' denotes that the trace over the fermion zero modes has been carried out. Moreover long supermultiplets carry additional fermion zero modes so they won't be captured by  $B_6$ .

The index  $B_6$  should be U-duality invariant. This translates to <sup>2</sup>

$$B_6(\Gamma, \phi_\infty) = B_6(\Gamma', \phi'_\infty) \quad (2.4)$$

where both  $\Gamma$  and  $\Gamma'$  and  $\phi_\infty$  and  $\phi'_\infty$  are related by a  $G(\mathbb{Z})$  transformation. If two dyons belong to the same duality orbit, immediately we know that they have the same index. In the problem of microstate counting it is important to identify duality orbits through charge invariants.

Both the electric and magnetic charge vectors live in a  $\Gamma^{22,6}$  Narain lattice from which we can construct the continuous T-duality invariants

$$Q^2 = Q^T L Q, \quad P^2 = P^T L P, \quad Q \cdot P = Q^T L P \quad (2.5)$$

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<sup>2</sup>Note this equation is only valid in a region of the moduli space. At special codimension one regions the index can jump. Phenomena also known as wall-crossing.

with  $L$  the  $SO(22, 6, \mathbb{R})$  invariant metric.

One important continuous U-duality invariant is the quartic Cremmer-Julia invariant

$$\Delta(\Gamma) = \det(\Gamma\Gamma^T) = Q^2 P^2 - (Q.P)^2. \quad (2.6)$$

Later we will see that the entropy is proportional to  $\sqrt{\Delta}$ . Because the U-duality group is discrete we can have more interesting invariants. One of major importance in the characterization of duality orbits is the arithmetic invariant [31]

$$I = \gcd(Q \wedge P) = \gcd(Q_i P_j - Q_j P_i). \quad (2.7)$$

This invariant will play an important role in the counting of 1/4-BPS dyons.

## 2.2 Duality orbits and invariants

As mentioned before,  $\mathcal{N} = 4$  string theory has U-duality symmetry

$$G(\mathbb{Z}) = SL(2, \mathbb{Z}) \times SO(22, 6; \mathbb{Z}) \quad (2.8)$$

composed of S and T-duality symmetries. As a consequence the index should be invariant under U-duality transformations of the charge vectors.

Under a rotation  $\Omega \in SO(22, 6; \mathbb{Z})$ , the charge vectors transform as

$$Q \rightarrow \Omega Q, P \rightarrow \Omega P, \quad (2.9)$$

while the Lorentzian metric  $L$  and the Narain lattice  $\Lambda$  are left invariant

$$\Omega^T L \Omega = L, \Omega \Lambda = \Lambda. \quad (2.10)$$

As mentioned before we can construct the T-duality invariants  $Q^2$ ,  $P^2$  and  $Q.P$  which are left invariant under the continuous  $G(\mathbb{R}) \supset G(\mathbb{Z})$  U-duality group. Additional discrete invariants can be constructed. These are necessary to completely characterize a T-duality orbit.

Consider a dyon with primitive  $(Q, P)$  charge vectors, that is, a dyon that cannot fragment into "smaller" dyons. This means that the charge vector cannot be written as multiple of a  $(Q_0, P_0)$  vector but it doesn't imply that the electric and magnetic charge vectors have to be individually primitive. We can represent these charge vectors in a sublattice  $\Lambda_0$  generated by  $e_1, e_2$  as

$$Q = r_1 e_1, P = r_2(u_1 e_1 + r_3 e_2), r_1, r_2, r_3, u_1 \in \mathbb{Z}^+ \quad (2.11)$$

such that  $\gcd(r_1, r_2) = \gcd(r_3, u_1) = 1$  and  $1 \leq u_1 \leq r_3$ . Recent work on the classification of  $SO(22, 6; \mathbb{Z})$  T-duality invariants [49] allows the identification of the set of integers

$$Q^2, P^2, Q.P, r_1, r_2, r_3 \text{ and } u_1 \quad (2.12)$$

as the complete set of T-duality invariants.

In these variables the discrete U-duality invariant  $I$  becomes  $r_1 r_2 r_3$ . This means that for a primitive dyon, that is, for a dyon with  $I = 1$ ,  $r_1 = r_2 = r_3 = u_1 = 1$  and therefore the orbit becomes labelled by  $Q^2$ ,  $P^2$  and  $Q.P$  only. As a matter of fact the partition function for a primitive dyon depends only on the continuous invariants. For non-primitive dyons it is expected the index  $B_6$  to have non trivial dependence on  $I$  and the remaining integers.

We can also explore the consequence of S-duality on these integers. It was shown in [50] that the set  $(r_1, r_2, r_3, u_1)$  can be brought to the form  $(I, 1, 1, 1)$  by an  $SL(2, \mathbb{Z})$  transformation. The charge vector acquires a much simpler representation

$$Q = Ie'_1, \quad P = e'_1 + e'_2. \quad (2.13)$$

In this new "frame" the derivation of the dyon partition function becomes easier since most of the invariants are trivial. Moreover the set  $(I, 1, 1, 1)$  is left invariant under the action of a subgroup  $\Gamma^0(I)$  of  $SL(2, \mathbb{Z})$  and therefore we expect the index  $B_6$  to exhibit this symmetry explicitly. The subgroup  $\Gamma^0(I)$  is defined by matrices

$$\begin{pmatrix} a \bmod(I) & \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

### 2.3 The dyon partition function

The Siegel modular form  $\Phi_{10}$  is for 1/4-BPS dyons as the ramanujan function  $\Delta = \eta^{24}$  is for 1/2-BPS states. The first is a modular form of  $Sp(2, \mathbb{Z})$ , the modular group of genus two riemann surfaces, while the second is the lower dimensional version, that is, for genus one surfaces. Using this analogy and consistency with electric and magnetic duality, lead Dijkgraaf, Verlinde and Verlinde [45] long time ago to propose  $\Phi_{10}^{-1}$  as the dyon partition function. This clue was remarkable and many other works followed in its derivation [51, 46, 22, 24].

In the work [26], which we review next, the authors gave a detailed derivation of the dyon partition function from first principles. Nevertheless only primitive dyons were concerned. Later it was shown in [31] that the discrete invariant  $I$  plays a non-trivial role in the counting. In [29, 33] the authors consider the case  $I > 1$  and propose a degeneracy formula based on duality symmetries and consistency checks much like Dijkgraaf, Verlinde and Verlinde did. Following this proposal, in [30] we attempt to give a physical sigma model interpretation of that result.

#### 2.3.1 Primitive dyons: $I = 1$

Also known as Igusa cusp form,  $\Phi_{10}$  is the unique weight 10 form of  $Sp(2, \mathbb{Z})$ . It depends on three complex numbers which encode the modular parameters of a genus two riemann surface. They can be packaged in a symmetric two dimensional matrix

$$\tau = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$$

taking values in the Siegel upper half plane, defined as

$$\text{Im}(\rho) > 0, \text{Im}(\sigma) > 0, \text{Im}(\rho)\text{Im}(\sigma) - \text{Im}(v)^2 > 0. \quad (2.14)$$

Under a transformation

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbb{Z})$$

with  $A, B, C, D$   $2 \times 2$  matrices, the matrix  $\tau$  transforms as

$$\tau \rightarrow \tau' = (A\tau + B)(C\tau + D)^{-1} \quad (2.15)$$

in analogy with  $Sp(1)$  modular transformations in a torus. Correspondingly  $\Phi_{10}$  shows the modular property

$$\Phi_{10}(\tau') = \det(C\tau + D)^{10} \Phi_{10}(\tau). \quad (2.16)$$

The subgroup  $SL(2, \mathbb{Z})$  can be realized in  $Sp(2, \mathbb{Z})$  via matrices of the form

$$g = \begin{pmatrix} A^T & 0 \\ 0 & A^{-1} \end{pmatrix} \text{ with } A \in SL(2, \mathbb{Z})$$

As can be easily checked this transformation leaves  $\Phi_{10}$  invariant. As explained before,  $SL(2, \mathbb{Z})$  invariance of the index concerns the set  $(r_1, r_2, r_3, u_1) = (1, 1, 1, 1)$ , that is, of primitive dyons.

The index is extracted performing an inverse fourier transform of  $\Phi_{10}^{-1}$

$$B_6(\Gamma, \phi_\infty) = (-1)^{Q \cdot P + 1} \int_{\mathcal{C}(\phi_\infty)} d^3\tau \frac{e^{-\pi i \Gamma^T \tau \Gamma}}{\Phi_{10}(\tau)}. \quad (2.17)$$

where the integration goes over a three dimensional torus

$$0 \leq \text{Re}(\rho) \leq 1, \quad 0 \leq \text{Re}(\sigma) \leq 1, \quad 0 \leq \text{Re}(v) \leq 1 \quad (2.18)$$

at fixed large values of the imaginary part of  $\tau$

$$\text{Im}(\rho) \gg 1, \quad \text{Im}(\sigma) \gg 1, \quad \text{Im}(v) \gg 1. \quad (2.19)$$

This defines the integration contour  $\mathcal{C}$ . Note the dependence of the integration contour on the moduli space measured at infinity  $\phi_\infty$ . Later we show that this dependence can lead to wall crossing. As expected from the analysis of duality orbits of  $I = 1$  the index shows dependence on only  $Q^2$ ,  $P^2$  and  $Q \cdot P$  via  $\Gamma^T \tau \Gamma = Q^2 \rho + P^2 \sigma + 2Q \cdot P v$ .

### 2.3.2 Derivation from physical grounds

This section is based on [26] which we review in the following.

Without loosing generality we can restrict to a charge sub-lattice  $\Gamma^{2,2} \subset \Gamma^{22,6}$  corresponding to the reduction on a particular two-torus  $T^2 = S^1 \times \tilde{S}^1$ . In this sector we have four electric and four magnetic charges. The charge configuration is taken be of the form

$$\Gamma = \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \tilde{n} & n & \tilde{w} & w \\ W & \tilde{W} & \tilde{K} & K \end{bmatrix}_H.$$

where the indice  $H$  denotes the heterotic frame. The charges  $n, \tilde{n}$  denote momentum on the circles  $S^1$  and  $\tilde{S}^1$  respectively while  $w, \tilde{w}$  stand for winding charges on the respective circles. The magnetic charges  $W, \tilde{W}$  correspond to NS5-branes wrapped on  $S^1 \times T^4$  and  $\tilde{S}^1 \times T^4$  respectively. Additionally we can have Kaluza-Klein monopoles  $K, \tilde{K}$  associated with the circles  $S^1$  and  $\tilde{S}^1$  respectively. We endow the lattice  $\Gamma^{2,2}$  with a metric  $L$  invariant under  $SO(2, 2, \mathbb{Z})$ ,

$$L = \begin{pmatrix} 0_{2 \times 2} & \mathbf{1}_{2 \times 2} \\ \mathbf{1}_{2 \times 2} & 0_{2 \times 2} \end{pmatrix}.$$

With this metric we construct the T-duality invariants

$$Q^2 = 2(\tilde{n}\tilde{w} + nw), \quad P^2 = 2(W\tilde{K} + \tilde{W}K), \quad Q.P = \tilde{n}\tilde{K} + nK + W\tilde{w} + \tilde{W}w \quad (2.20)$$

In the presence of NS5-branes the microscopic theory is strongly coupled and there's not much information we can extract. We avoid this problem by going to the IIB frame and consider a system of D-branes coupled to KK monopoles where a weakly coupled description is available. We perform the following chain of dualities. A string-string duality maps Heterotic string on  $T^4 \times S^1 \times \tilde{S}^1$  to IIA on  $K3 \times S^1 \times \tilde{S}^1$  which is further T-dualized to give IIB on the dual circle  $\hat{S}^1$  and finally we do a ten dimensional S-duality. Lets see more carefully what is happening to the charges under this chain of transformations.

1. **Six dimensional string-string duality, Het to IIA:** the momentum and kaluza klein charges don't transform while the Poincaré electric-magnetic duality of the six dimensional NS-NS B field takes winding charge to NS5-brane charge and vice-versa.

$$\Gamma = \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \tilde{n} & n & \tilde{W} & W \\ w & \tilde{w} & \tilde{K} & K \end{bmatrix}_{IIA}.$$

2. **T-duality along  $\tilde{S}^1$ , IIA to IIB:** this duality maps IIA on the circle  $\tilde{S}^1$  to IIB on the dual circle  $\hat{S}^1$ . The momentum and winding charges associated with this circle are exchanged. The same happens for NS 5-branes and KK monopoles.

$$\Gamma = \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \tilde{w} & n & \tilde{W} & \tilde{K} \\ w & \tilde{n} & W & K \end{bmatrix}_{IIB}.$$

3. **Ten dimensional S-duality, IIB to IIB:** this transformation maps winding charges to D1-branes and NS 5-branes to D5-branes. Other charges remain untouched.

$$\Gamma = \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \tilde{Q}_1 & n & \tilde{Q}_5 & \tilde{K} \\ Q_1 & \tilde{n} & Q_5 & K \end{bmatrix}_{IIB}.$$

Here  $Q_1$  and  $\tilde{Q}_1$  represent charges associated with D1-branes wrapping a circle  $S^1$  and  $\hat{S}^1$  respectively. Analogously  $Q_5$  and  $\tilde{Q}_5$  represent D5-branes wrapping  $K3 \times S^1$  and  $K3 \times \hat{S}^1$ .

For simplicity we take a charge configuration of the form

$$\Gamma = \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} 0 & n & 0 & \tilde{K} \\ Q_1 & J & Q_5 & 0 \end{bmatrix}_H.$$

which corresponds to a system of  $Q_1$  D1-branes and  $Q_5$  D5-branes wrapping  $S^1$  and  $K3 \times S^1$  respectively in the background of  $\tilde{K}$  KK-monopoles, with momentum  $n$  and  $J$  along the circles  $S^1$  and  $\tilde{S}^1$ . This configuration is also known as D1-D5-KK system. If we impose primitivity on the charge vectors we get the following condition

$$I = \gcd(Q \wedge P) = \gcd(Q_1 n, Q_1 \tilde{K}, n Q_5, J \tilde{K}, Q_5 \tilde{K}) = 1 \quad (2.21)$$

which can be satisfied imposing  $\tilde{K} = 1$  and  $\gcd(Q_1, Q_5) = 1$ . The general case with arbitrary number of KK monopoles will be studied later for non-primitive dyons. The condition  $\gcd(Q_1, Q_5)$  is known to be a physical requirement for the existence of D1-D5 bound states at threshold [32, 27, 28].

In weak coupling limit both the D-branes and the KK-monopole are weakly interacting. We can see the D1-D5 brane system moving as a particle in the transverse four dimensional Taub-Nut (TN) geometry which is the solution of Einstein's equations in the presence of a Kaluza-Klein monopole. The ten dimensional geometry is

$$ds^2 = -dt^2 + ds_{Taub-Nut}^2 + ds_{K3 \times S^1}^2 \quad (2.22)$$

with the Taub-Nut metric given by

$$ds_{Taub-Nut}^2 = \left(1 + \frac{R}{r}\right) (dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2) + R^2 \left(1 + \frac{R}{r}\right)^{-1} (2d\psi + \cos(\theta) d\phi)^2 \quad (2.23)$$

The TN space has the particularity that near the origin  $r = 0$  it looks like  $\mathbb{R}^4$  while for large  $r$  it asymptotes to  $\mathbb{R}^3 \times \tilde{S}^1$ . From the point of view of the observer at infinity he sees a theory in four-dimensions. The TN geometry possesses in addition a normalizable 2-form  $\omega_{Taub-Nut}^{(2)}$ .

The microscopic theory can be described by three weakly interacting parts. Each of these can be realized as a two dimensional supersymmetric sigma model on  $\mathbb{R}^t \times S^1$  [26]. We denote the weakly interacting parts as

1. **Higgs branch of D1-D5:** describes the moduli space of vacua of the low energy theory of the D1-D5 brane system on  $K3$ . In the Higgs branch [52] the (1,5) strings acquire vevs forcing the D1 and D5 branes to sit on top of each other. In the IR, the low energy theory is described by a two dimensional SCFT with sigma model the Hilbert scheme of  $Q_1 Q_5 + 1$  points on  $K3$  which is isomorphic to the symmetric product of  $K3$  at the orbifold point

$$\mathcal{M}_{D1-D5} = \mathbf{Sym}^{Q_1 Q_5 + 1}(K3) \quad (2.24)$$

This is a (4,4) SCFT with R-symmetry  $SU(2)_L \times SU(2)_R$ . The R-symmetry corresponds to rotations in the transverse space.

2. **Center of mass motion of the D1-D5 system:** it describes the vector multiplet degrees of freedom of (1,1) and (5,5) strings. We can see the D1-D5 as a particle moving in the TN space. From the motion on TN we have 4 scalars transforming under the vector representation of  $SU(2)_L \times SU(2)_R$  and 4 left-moving and 4 right-moving fermions transforming in the fundamental of  $SU(2)_R$  and  $SU(2)_L$  respectively. The Taub-Nut background breaks half the susy's. This gives rise to a (0,4) SCFT on  $\mathbb{R}^4$ . The sigma model has target space

$$\mathcal{M}_{CM} = \mathbb{R}^4 \quad (2.25)$$

This target space contrasts with the curved TN. We note that the index is a quantity that doesn't depend on the parameters of the theory. We use this property to compute the index for very large  $R$  radius which is equivalent as putting the D1-D5 at  $r = 0$ , where TN looks like  $\mathbb{R}^4$ . This comment fails for the case of zero modes where we need to be more careful.

3. **KK monopole closed string excitations:** this describes the low energies excitations of closed strings in the Taub-Nut background. We have 3 massless scalars coming from the breaking of  $\mathbb{R}^3$  translation. Additionally the reduction of the Ramond-Ramond C 4-form on  $\omega_{Taub-Nut}^{(2)} \wedge \omega_{K3}^{(2)}$  gives 19 left-moving scalars and 3 right-moving scalars. Additionally the NS-NS B field and the Ramond-Ramond 2-form give together 2 extra scalars. In total we have 24 left-moving and 8 right-moving scalars. The Taub-Nut preserves half susy's of IIB on  $K3$ . This gives in addition 8 right-moving fermions. We denote the resulting sigma model by  $\sigma_{KK}$ .

The analysis of the zero modes of the supersymmetric field theory on Taub-Nut requires special care. The dynamics is of a  $\mathcal{N} = 4$  superparticle with 4 bosonic and 4 fermionic coordinates moving in the Taub-Nut space. So far we have been working in a point in the moduli space where  $S^1$  and  $\tilde{S}^1$  are orthogonal. A mixing between the circles can be achieved by a  $d\psi + ady$  translation. As a consequence the tension of D1-D5 brane system  $\sqrt{g_{yy}}$  generates a potential

$$V(r) = a^2 R^2 \left(1 + \frac{R}{r}\right)^{-1} \quad (2.26)$$

which under supersymmetrization originates other fermionic terms. Under this potential supersymmetric bound states can form and contribute to the total index.

Concerning the fermionic zero modes resulting from the broken susy's, the analysis goes as follows. IIB string theory on  $K3 \times \text{TN}$  preserves 8 left-moving susy's on the 1 + 1 world volume theory. The breaking of 8 supersymmetries gives rise to 4 complex fermion zero modes. Additionally the D1-D5 breaks 4 of the remaining 8 susy's contributing with 2 complex fermion zero modes. This gives a total of 6 complex fermion zero modes as expected for a 1/4-BPS dyon.

We now proceed to the construction of index.



We use the index  $B'_6 = \text{Tr}(-1)^{2h}$  where tracing over fermion zero modes has been carried out. We find convenient to compute the generating function, also known as elliptic genus,

$$\chi(q, \bar{q}, y, \tilde{y}; \mathcal{M}) = \text{Tr}_{\text{R-R}}(-1)^{2J_L - 2J_R} q^{L_0} \bar{q}^{\bar{L}_0} y^{2J_L} \tilde{y}^{2J_R} \quad (2.27)$$

with

$$q = e^{2\pi i \rho}, \quad y = e^{2\pi i v}, \quad \tilde{y} = e^{2\pi i \tilde{v}}, \quad (2.28)$$

which corresponds to the partition function of the sigma model with Ramond-Ramond boundary conditions. The generators  $L_0$  and  $\bar{L}_0$  are the usual left and right Virasoro dilatation generators while  $J_L$  and  $J_R$  correspond to the Cartan generators of  $SU(2)_L \times SU(2)_R$ , the little group in five dimensions. We contrast this with the little group in four dimensions which is  $SO(3)$ . Due to the particular fibration structure of the TN space (2.23) there is an interesting connection between five and four dimensional black holes known as 4d-5d lift [46]. While at the tip of TN space the geometry looks like  $\mathbb{R}^4$ , at asymptotic infinity it looks like  $\mathbb{R}^3 \times \tilde{S}^1$ . If now we put the D1-D5 system at the tip of the TN, the transverse space looks five dimensional. Therefore we can relate the degrees of freedom of the five dimensional BMPV black hole [53] to the D1-D5-KK four dimensional black hole. The rotation generator  $J_L$  measured at  $r = 0$  is further identified with  $U(1)$  translations on the circle  $\tilde{S}^1$  at asymptotic infinity.

Due to supersymmetry, the right movers are forced to stay in the ground state and therefore the dependence of the function  $\chi$  on  $\bar{q}$  drops meaning that only BPS states are being counted. We now show the different contributions to the generating function:

### 1. Higgs branch of D1-D5:

$$\sum_{N=0}^{\infty} p^{N-1} \chi(q, y; \text{Sym}^N(K3)) = \frac{1}{p} \prod_{n \geq 1, m \geq 0, l \in \mathbb{Z}} \frac{1}{(1 - p^n q^m y^l)^{c(4nm - l^2)}} \quad (2.29)$$

with  $c(n)$  defined via the equation

$$\chi(q, y; K3) = 8 \left[ \frac{v_2(\tau, z)^2}{v_2(\tau, 0)^2} + \frac{v_3(\tau, z)^2}{v_3(\tau, 0)^2} + \frac{v_4(\tau, z)^2}{v_4(\tau, 0)^2} \right] = \sum_{n, j \in \mathbb{Z}} c(4n - j^2) e^{2\pi i n + 2\pi i j z} \quad (2.30)$$

### 2. CM contribution:

$$\chi(q, y; \mathbb{R}^4) = \frac{\prod_{n \geq 1} (1 - q^n)^4}{\prod_{n \geq 1} (1 - q^n y)^2 (1 - q^n y^{-1})^2} \quad (2.31)$$

### 3. KK closed string excitations:

$$\chi(q, \sigma_{KK}) = \text{Tr}(-1)^F q^{L_0} \bar{q}^{\bar{L}_0} = \frac{1}{q} \frac{1}{\prod_{n \geq 1} (1 - q^n)^{24}} \quad (2.32)$$

This is a four dimensional index. States don't carry charge  $\tilde{n}$  and dependence on  $y$  drops. This index is the same as the 1/2-BPS index that counts electric states in the heterotic string. In fact the system KK-P can be mapped to Heterotic momentum-winding states using duality symmetry.

#### 4. $\mathcal{N} = 4$ superparticle in Taub-Nut:

$$\text{Tr}(-1)^F y^{\tilde{J}} = \sum_{j \geq 1} j e^{2\pi i j v} = \frac{e^{2\pi i v}}{(1 - e^{2\pi i v})^2} \quad (2.33)$$

where  $\tilde{J}$  is the momentum charge on the circle  $\tilde{S}^1$ . Note that the last expression can be expanded either in powers of  $e^{2\pi i v}$  or  $e^{-2\pi i v}$ . This generates ambiguity when trying to extract the fourier coefficient. We show later that this is related to wall-crossing.

Putting all factors together we get

$$-\frac{1}{pqy} \prod_{\substack{n,m \geq 0, l \in \mathbb{Z} \\ l < 0 \text{ for } k=l=0}} \frac{1}{(1 - p^n q^m y^l)^{c(4nm-l^2)}} \quad (2.34)$$

with

$$p = e^{2\pi i \sigma}, \quad q = e^{2\pi i \rho}, \quad y = e^{2\pi i v} \quad (2.35)$$

which is equal to  $-\Phi_{10}^{-1}$ .

The chemical potentials  $\rho$ ,  $\sigma$  and  $v$  couple to  $P^2 = 2Q_1 Q_5$ ,  $Q^2 = 2N$  and  $Q.P = \tilde{J}$  respectively. To extract the index  $B_6$  we perform an inverse fourier transform

$$B_6(Q^2, P^2, Q.P) = (-1)^{Q.P+1} \int_{\mathcal{C}} d\rho d\sigma dv \frac{e^{-i\pi\rho Q^2 - i\pi\sigma P^2 - 2\pi i v Q.P}}{\Phi_{10}(\rho, \sigma, v)} \quad (2.36)$$

where the contour  $\mathcal{C}$  is as given in (2.18, 2.19). The additional factor  $(-1)^{Q.P}$  is reminiscent of going from five to four dimensions [46, 23].

#### 2.3.3 Consistency checks

In the limit of large charges  $Q^2 \gg 1$ ,  $P^2 \gg 1$  and  $Q.P \gg 1$  we can make an asymptotic expansion of  $B_6$  (2.36). The leading term can then be compared with the black hole entropy valid in the same limit. The idea is to deform the contour  $\mathcal{C}$  such that it passes near a pole whose residue contribution is much leading than the left over integral [45, 26, 54]. The Siegel form  $\Phi_{10}$  has second order zeros at

$$n_2(\rho\sigma - v^2) + bv + n_1\sigma - m_1\rho + m_2 = 0, \quad (2.37)$$

with  $n_1, n_2, m_1, m_2 \in \mathbb{Z}$  and  $b \in 2\mathbb{Z} + 1$  obeying the condition  $n_1 m_1 + n_2 m_2 + b^2/4 = 1/4$ . The residue at  $(n_1, n_2, m_1, m_2, b) = (1, 0, 0, 0, 1)$ , modulo  $SL(2, \mathbb{Z})$  transformations, gives the leading term in the asymptotic expansion which is the correct result for the entropy

$$B_6(Q^2, P^2, Q.P \gg 1) \approx e^{\pi\sqrt{\Delta(Q,P)}} + \mathcal{O}(e^{\pi\frac{\sqrt{\Delta}}{2}}) \quad (2.38)$$

Subleading perturbative corrections to the microscopic answer can be computed. In fact for sufficiently large charges we can approximate  $B_6$  by

$$B_6 \approx K_0 (-1)^{Q.P} \int \frac{d^2\tau}{\tau_2^2} \left( 26 + \frac{\pi}{\tau_2} |Q + \tau P|^2 \right) e^{\frac{\pi}{2\tau_2} |Q + \tau P|^2 - 24 \ln \eta(\tau) - 24 \ln \eta(-\bar{\tau}) - 12 \ln(\tau_2)} \quad (2.39)$$

which in the saddle point approximation reduces to (2.38). Subleading non-perturbative corrections are suggestive of multi-center black hole contribution [54].

The residues for  $n_2 = 0$  are even more subleading. They encode phenomena associated with wall-crossing. Although the integrand in (2.36) is manifestly  $SL(2\mathbb{Z})$  invariant the contour is not. After such transformation we may cross a pole in deforming the contour to its original form. It happens that only a pole  $n_2 = 0$  can be crossed in the deformation. Take for example the residue at  $(n_1, n_2, m_1, m_2, b) = (0, 0, 0, 0, 1)$  which corresponds to the pole  $v = 0$ . Near this pole the partition function behaves like

$$\frac{1}{\Phi_{10}} \approx \frac{1}{v^2 \eta^{24}(\rho) \eta^{24}(\sigma)} \quad (2.40)$$

In this case the index jumps by the amount

$$\Delta B_6 = \text{Res}_{v=0} = (-1)^{Q \cdot P} (Q \cdot P) \int \frac{e^{-\pi i \rho Q^2}}{\eta^{24}(\rho)} \int \frac{e^{-\pi i \sigma P^2}}{\eta^{24}(\sigma)} \quad (2.41)$$

In a different context, this can be easily recognized as the Denef's split attractor formula for 1/2-BPS black holes in  $\mathcal{N} = 2$  supergravity [39, 43, 44]

$$\Delta \Omega = (-1)^{\langle \Gamma^1, \Gamma^2 \rangle} \langle \Gamma^1, \Gamma^2 \rangle \Omega(\Gamma^1) \Omega(\Gamma^2) \quad (2.42)$$

with  $\langle \Gamma^1, \Gamma^2 \rangle = Q \cdot P$  and  $\Omega(\Gamma)$  is the index that counts 1/2-BPS states. In  $\mathcal{N} = 4$  string theory the index that counts 1/2 BPS states is [55]

$$d_{1/2}(Q^2) = \int \frac{e^{-\pi i \rho Q^2}}{\eta^{24}(\rho)} \quad (2.43)$$

From a microscopic point of view we can understand the jump in the index as the decay of a 1/4-BPS dyon into its 1/2-BPS constituents, that is,

$$(Q, P) \rightarrow (Q, 0) + (0, P). \quad (2.44)$$

From the gravity point of view it corresponds to the appearance or disappearance of a two center black hole [56]. In fact for large charges the jump in the index can be interpret as coming from the contribution of two centers which are very far from each other

$$\ln(\Delta B_6) \approx 2\pi\sqrt{Q^2} + 2\pi\sqrt{P^2}. \quad (2.45)$$

Other poles with  $n_2 = 0$  correspond to more complex decays which are basically related by a  $SL(2, \mathbb{Z})$  transformation to the  $v = 0$  case. For more details we refer the reader to [40].

Physically the picture is the following. The  $SL(2, \mathbb{Z})$  transformation acts not just on the charges but also on the axion-dilaton  $\lambda_\infty$ . Since the mass has a non-trivial dependence on  $\lambda_\infty$  it will change as we move on the moduli space. When the mass of the quarter-BPS dyons equals the sum of the masses of the half-BPS dyons for  $\lambda_*$ , that is,

$$m_{1/4-BPS}(Q, P)|_{\lambda_*} = m_{1/2-BPS}(Q) + m_{1/2-BPS}(P) \quad (2.46)$$

it becomes marginally stable and can decay into its 1/2-BPS constituents. The regions in the moduli space where the dyon becomes marginally stable are codimension one walls. Schematically we have the moduli space divided into chambers  $(X, X', X'', \dots)$  separated by codimension one walls. The index  $B_6$  is piecewise constant in these chambers.

Consider the example of a dyon with  $Q^2 = P^2 = -1$  and  $Q.P = j > 0$ . We can easily extract the index  $B_6$  from (2.33). It gives  $B_6 = (-1)^{j+1}j$ . Under a S-duality transformation

$$\begin{bmatrix} Q \\ P \end{bmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} Q \\ P \end{bmatrix}$$

the T-duality invariants are mapped to  $Q^2 = P^2 = -1$  and  $Q.P = -j < 0$ . This is equivalent to the change of contour  $\text{Im}v \rightarrow -\text{Im}v$ . In deforming the countour to its original value we pick a residue at  $v = 0$ . In this case the jump is easy to compute and formula 2.41 gives  $\Delta B_6 = (-1)^{j+1}j$ . At the same time the axion-dilaton gets transformed to  $-1/\lambda$  and the dyon jumps from one chamber to another separated by a wall at  $\text{Re}(\lambda) = 0$ . In this new chamber the index (2.33) only contains positive powers of  $e^{2\pi i v}$  which gives a zero index consistent with the predicted jump.

In [57] the authors propose a contour which captures only the contribution from single center black holes. In this case the index becomes moduli independent and therefore the dyon is free from decaying. The prescription is the following

$$\text{Im}(\rho) = \Lambda \left( \frac{|\lambda|^2}{\lambda_2} + \frac{Q_R^2}{\sqrt{\Delta_R}} \right) \quad (2.47)$$

$$\text{Im}(\sigma) = \Lambda \left( \frac{1}{\lambda_2} + \frac{P_R^2}{\sqrt{\Delta_R}} \right) \quad (2.48)$$

$$\text{Im}(v) = -\Lambda \left( \frac{\lambda_1}{\lambda_2} + \frac{Q_R.P_R}{\sqrt{\Delta_R}} \right) \quad (2.49)$$

with  $Q_R^2 = Q^T(M+L)Q$ ,  $P_R^2 = P^T(M+L)P$ ,  $Q_R.P_R = Q^T(M+L)P$  and  $\Delta_R = Q_R^2 P_R^2 - (Q_R.P_R)^2$ . The matrix  $M$  is a symmetric  $28 \times 28$  matrix which encodes the 132 moduli of the theory and obeys the constraint  $M^T L M = L$ , with  $L$  the metric on  $\Gamma^{6,22}$ . The scalar  $\lambda$  is the axion-dilaton. The parameter  $\Lambda$  is taken to be very large to ensure the dyon doesn't leave this chamber.

## 2.4 Non-primitive dyons: $I > 1$

Derivation of the spectrum of non-primitive dyons from physical grounds is more complex. As a matter of fact, it was noted long time ago that the counting of non-primitive charge vectors, in the context of toroidally compactified IIB string theory, was a difficult problem [27]. The case of the D1-D5 system with  $Q_1$  and  $Q_5$  not coprime is a good example. Since the system can split at no cost of energy, this signals the presence of singularities in the moduli space of the low energy theory [32].

In the case of 1/4-BPS dyons with non-primitive charge vectors, similar difficulties are encountered. Consider a charge configuration of the form

$$\Gamma = \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} 0 & nI & 0 & kI \\ Q_1 & J & Q_5 & 0 \end{bmatrix}_H. \quad (2.50)$$

with  $(Q_1, Q_5)$  coprime. We choose charges such that  $\gcd(Q \wedge P) = I > 1$ . In this case we have to consider a configuration multi KK-monopoles. If we were to repeat the analysis done in the  $I = 1$  case, we would face the following difficulties

1. Multi KK monopoles have collective coordinates which parametrize a non-trivial moduli space. The study of bound states in this background is a very difficult problem.
2. The multi KK geometry admits  $I$  non-trivial 2-cycles. For each pair of KK monopoles there is a 2-cycle that touches both of them [58]. The area of this 2-cycle is proportional to the distance between the two centers and approaches zero when the monopoles touch each other. A D3-brane wrapping such cycle will give rise to tension less strings [59]. In the counting we should consider a possible contribution from these strings.

Aware of these problems, the authors in [29, 33] proposed an index formula much as Dijkgraaf, Verlinde, Verlinde have made for the case of primitive dyons. This formula is consistent with many properties known for non-primitive dyons and corresponding black holes. The proposed index has the form

$$d(Q, P) = (-1)^{Q \cdot P + 1} \sum_{s|I} s^4 \int_{\mathcal{C}(s)} d^3\tau \frac{e^{-\pi i \Gamma^T \tau \Gamma}}{\Phi_{10}(\rho, s^2\sigma, sv)} \quad (2.51)$$

with contour

$$\mathcal{C}(s) : 0 \leq \text{Re}\rho \leq 1, 0 \leq \text{Re}\sigma \leq \frac{1}{s^2}, 0 \leq \text{Re}v \leq \frac{1}{s}. \quad (2.52)$$

After a simple manipulation we can write it in a more convenient form

$$d(Q, P) = \sum_{s|I} s d_1\left(\frac{Q^2}{s^2}, P^2, \frac{Q \cdot P}{s}\right) \quad (2.53)$$

where  $d_1(a, b, c)$  denotes the fourier coefficient extracted from the primitive answer (2.36). The main driving principle for the such construction is based on wall crossing for non-primitive decay. In the case of a primitive decay<sup>3</sup>, there is a one to one correspondence between the decay and the pole in  $\Phi_{10}^{-1}$ . That is, take the most general primitive decay

$$(Q, P) \rightarrow (\alpha Q + \beta P, \gamma Q + \delta P) + (\delta Q - \beta P, -\gamma Q + \alpha P) \quad (2.54)$$

with  $\alpha\delta = \gamma\beta$  and  $\alpha + \delta = 1$ . The set of integers  $(\alpha, \beta, \gamma, \delta)$  gives the location of the pole at  $\rho\gamma - \sigma\beta + v(\alpha - \beta) = 0$ .

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<sup>3</sup>A primitive decay is one for which the products of the decay are primitive dyons.

Once  $I > 1$  a marginally stable dyon can decay into products which are non-primitive. This allows for a larger set of integers  $(\alpha, \beta, \gamma, \delta)$ . In [33] they postulate that such correspondence should remain even in the non-primitive case. This was helpful in suggesting part of the pole structure of the partition function which is indeed that of (2.51).

Take the example of a non-primitive decay

$$(IQ_0, P) \rightarrow (IQ_0, 0) + (0, P). \quad (2.55)$$

associated with the pole at  $v = 0$ . The wall crossing formula extracted from the residue of (2.51) gives a jump of the form

$$\Delta d(Q, P) = (-1)^{Q \cdot P} (Q \cdot P) \sum_{s|I} d_{1/2}(Q^2/s^2) d_{1/2}(P^2) \quad (2.56)$$

Again for large charges, the term  $s = 1$  in (2.51) gives the leading contribution reproducing correctly the black hole entropy

$$d(Q, P) \approx \sum_{s|I} e^{S_{BH}/s}. \quad (2.57)$$

with  $S_{BH} = \pi\sqrt{\Delta}$ .

One additional requirement is the invariance of the index under  $\Gamma^0(I) \in SL(2, \mathbb{Z})$ . S-duality invariance demands that  $\Gamma'^T \tau \Gamma' = \Gamma^T \tau' \Gamma$  with  $\tau' = (h^T)^{-1} \tau h^{-1}$  and  $h \in \Gamma^0(I)$ . By embedding this subgroup in  $Sp(2, \mathbb{Z})$  via

$$g = \begin{pmatrix} (h^T)^{-1} & 0 \\ 0 & h \end{pmatrix} \text{ with } h \in \Gamma^0(I)$$

we can show that the integrand (2.51) is left invariant due to

$$\Phi_{10}(\rho', s^2 \sigma', s v') = \Phi_{10}(\rho, s^2 \sigma, s v). \quad (2.58)$$

There is yet another important check to this formula. At special points in the moduli space of the heterotic string on  $T^6$  we can have enhanced gauge symmetry. Away from these points the symmetry is spontaneously broken and the moduli fields play the role of the Higgs field. If their vevs are small the symmetry breaking scale is small compared to the string scale. In this case the theory contains states with masses much smaller than massive string states and should be part of the dyon spectrum. In particular it should include 1/4-BPS dyons of  $\mathcal{N} = 4$  SYM.

Dyon charges in  $SU(N)$  gauge theory are labelled by N-dimensional root vectors labelled by a pair  $(q, p)$ . It can be shown that a primitive embedding of the root lattice in the Narain lattice is possible. This means that

$$I = \gcd(Q \wedge P) = \gcd(q \wedge p). \quad (2.59)$$

Moreover the T-duality metric  $L$  is the negative of the Cartan metric which gives the following assignments

$$q \cdot q = -Q^2, \quad p \cdot p = -P^2, \quad q \cdot p = -Q \cdot P \quad (2.60)$$

Additionally we have  $q^2, p^2 \geq 0$  and  $(q.p)^2 \leq (q^2 + p^2)/2$  because the Cartan metric is positive definite and therefore it implies  $Q^2, P^2 < 0$  and  $(Q.P)^2 < -(Q^2 + P^2)/2$ .

Counting BPS dyons in SYM is the problem analysed in [60]. The authors computed an index  $\mathcal{I}$  in the gauge theory for dyons with torsion  $r = \gcd(q \wedge p)$  and found

$$\mathcal{I} = r. \quad (2.61)$$

In terms of string theory dyons the conditions mentioned above imply  $Q^2/2 = -I^2$ ,  $P^2 = -1$  and  $Q.P = \pm I$ . Neglecting issues of chamber dependence, formula (2.53) gives for these dyons  $d(Q, P) = I$  which agrees with (2.61).

### 2.4.1 Proposed sigma model from physical grounds

The geometry associated with  $I$  KK monopoles is the generalization of (2.23) to include multi centers

$$ds^2 = V^{-1}(dx^4 + \vec{\omega} \cdot d\vec{x})^2 + V d\vec{x} \cdot d\vec{x} \quad (2.62)$$

where  $x^4$  is a compact direction and  $\vec{x}$  is the position in  $\mathbb{R}^3$ . The harmonic function  $V$  and the connection  $\vec{\omega}$  are defined as

$$V = 1 + \sum_{s=1}^I V_s, \quad \vec{\omega} = \sum_{s=1}^I \vec{\omega}_s \quad (2.63)$$

$$V_s = \frac{4r}{|\vec{x} - \vec{x}_s|}, \quad \vec{\nabla} \times \vec{\omega}_s = \nabla V_s \quad (2.64)$$

At asymptotic infinity, when  $|\vec{x}|$  is very large, the geometry looks like  $\mathbb{R}^3 \times \tilde{S}^1$ . The moduli  $\vec{x}_s$  denote the position of the each of the  $I$  monopoles. If we zoom very close to one center the geometry looks like  $\mathbb{R}^4$  given that  $x^4$  has periodicity  $16\pi r$  to avoid a conical singularity.

In order to preserve supersymmetry we should consider the case when all the monopoles sit on top of each other.

In this case the geometry looks like that of a single monopole with charge  $I$  but with a conical singularity at the origin. The TN space becomes an asymptotically locally euclidean space (ALE)  $\mathbb{C}^2/\mathbb{Z}_I$ . The subgroup  $\mathbb{Z}_I$  is embedded in  $SU(2)_L$  of the tangent group  $SU(2)_L \times SU(2)_R$  defined at the origin, preserving this way the same number of supercharges as a single KK monopole<sup>4</sup>. At asymptotic infinity the radius of  $\tilde{S}^1$  is measured in units of  $1/I$  due to the orbifold. This implies that an asymptotic observer will measure a total momentum charge which is a multiple of  $I$ . This is consistent with the fact that  $Q.P$  is a multiple of  $I$  (2.50).

This instructs us to study the D1-D5 in transverse  $\mathbb{C}^2/\mathbb{Z}_I$  which is the problem studied in [61]. The author uses the standard approach of supersymmetric gauge field theory in ALE spaces [62]. We start by going to the covering space of  $\mathbb{C}^2/\mathbb{Z}_I$  which means enhancing the gauge group  $U(N_1) \times U(N_2) \times \dots$  of the gauge theory in  $\mathbb{C}^2$  to  $U(IN_1) \times U(IN_2) \times \dots$ . In our case the D1-D5 system in transverse  $\mathbb{C}^2$  is a supersymmetric gauge theory with gauge group

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<sup>4</sup>The holonomy group of TN is  $SU(2)_R$ , which means that it breaks half of the background supersymmetries

$U(Q_1) \times U(Q_5)$ . A careful analysis of the D-terms of the enhanced gauge theory reveals that the moduli space of vacua factorizes with some additional identifications. This last point is not clear in [61]. Denoting the moduli space of the D1-D5 on  $\mathbb{C}^2$  by  $\mathcal{M}_1$ , we propose

$$\mathcal{M}_I = \text{Sym}^I(\mathcal{M}_1)/\mathbb{Z}_I. \quad (2.65)$$

The identification under permutations comes from gauging the moduli space by the Weyl group  $S^I \subset U(IQ_1) \times U(IQ_5)$  left unbroken in the Higgs phase while the  $\mathbb{Z}_I$  orbifold comes from the breaking of the  $SU(2)_L \times SU(2)_R$  R-symmetry of the parent  $(4, 4)$  theory to  $U(1)_L \times SU(2)_R$ .

In other words we propose that the effective string that describes low energy fluctuations of the D1-D5 system on  $\mathbb{C}^2/\mathbb{Z}_I$  has a sigma model with target space

$$\sigma_I = \text{Sym}^I(\sigma(\mathbb{R})^4 \times \text{Sym}^{Q_1 Q_5 + 1}(K3)) / \mathbb{Z}_I \quad (2.66)$$

where  $\mathbb{Z}_I$  belongs to the  $SU(2)_L$  R-symmetry of the parent theory, that of D1-D5 on  $\mathbb{C}^2$ . This space is singular in contrast with  $I = 1$  case. Although we don't know how to resolve these singularities, the index is well defined and can be computed. As for  $I = 1$  the effective string is a  $(0, 4)$  SCFT.

Since we are interested in black holes which are not charged under  $\mathbb{Z}_I$  we look for states of the untwisted sector, that is, states of the parent  $(4, 4)$  theory invariant under  $\mathbb{Z}_I$ . This is equivalent to look for states that carry  $U(1)_L$  charge that is a multiple of  $I$ .

The sigma model carries two complex fermion zero modes, originally from the  $\mathbb{R}^4$  factor, which have to be symmetrized along with the other states. To correctly account for these we should use the appropriate helicity trace

$$B_2 = -\frac{1}{2} \text{Tr}(-1)^{2J_L - 2J_R} (2J_R)^2 \quad (2.67)$$

in the same spirit of (2.3). Because the zero modes are being symmetrized they have a non-trivial contribution to the index like in [27]. The application of the theorem of symmetrized products [63] gives

$$B_2(\sigma_I) = \sum_{s|I, s|n} s \hat{c}\left(Q_1 Q_5, \frac{nI}{s^2}, \frac{lI}{s}\right) \quad (2.68)$$

with the coefficients  $\hat{c}(a, b, c)$  defined via

$$B_2(\sigma_1) = \hat{c}(Q_1 Q_5, n, l) \quad (2.69)$$

which is the answer for the D1-D5 on  $\mathbb{C}^2$ . The charge  $n$  is the momentum along the circle  $S^1$  and  $l$  is the angular momentum of the black hole in five dimensions.

A derivation of (2.68) goes as follows. The Hilbert space of a symmetrized product can be decomposed as a sum labelled by partitions of  $I$

$$\mathcal{H}_I = \sum_{\sum k N_k = I} \prod_k \otimes S^{N_k}(\mathcal{H}_k) \quad (2.70)$$



with

$$S^N(\mathcal{H}) = \sum_{\text{permutations } \sigma} \epsilon(\sigma) \prod_N \otimes \mathcal{H}, \quad (2.71)$$

where  $\epsilon = (-1)^\sigma$  for fermionic states and  $\epsilon = 1$  for bosonic. The Hilbert space  $\mathcal{H}_k$  denotes a multiple wound string with size  $k$  such that  $\mathcal{H}_1$  is the effective string in the case  $I = 1$ .

Each of  $\mathcal{H}_k$  carries two complex fermion zero modes. Now a state  $|h_k\rangle \in S^{N_k}(\mathcal{H}_k)$  contributes to  $J_R$  with  $h_k$ . The operator  $(2J_R)^2$  in (2.67) becomes  $(2J_R)^2 = \sum_k (2h_k)^2 + \sum_{k \neq l} (2h_{r_k} 2h_{r_l})$  for a partition  $(N_k, k)$ . Due to the presence of fermion zero modes, the trace of  $(-1)^{2J_R} (2J_R)^2$  over  $\prod_k \otimes S^{N_k}(\mathcal{H}_k)$  is zero unless the partition obeys  $kN_k = I$ . In this case the index becomes

$$B_2 = \sum_{k \text{ with } kr=I} \text{Tr}_{S^r(\mathcal{H}_k)} (-1)^{2J-2h} (2h)^2. \quad (2.72)$$

A state in  $S^r(\mathcal{H}_k)$  is given by symmetric or antisymmetric wave functions  $\sum_\sigma \epsilon(\sigma) \prod_i |n_i, J_i, h_i\rangle$  if they are bosonic or fermionic accordingly. Each state carries total momentum  $\sum n_i = N$  and total angular momentum  $\tilde{J} = \sum 2J_i$ . Instead of computing directly the trace in (2.72), which is not trivial, we find it more convenient to compute the partition function first

$$f(\beta, q, y)_r = \text{Tr}_{S^r(\mathcal{H}_k)} (-1)^{\sum 2J_i - \sum 2h_i} e^{\beta \sum 2h_i} q^{\sum n_i} y^{\sum 2J_i}. \quad (2.73)$$

and then extract  $B_2$  by looking to a particular fourier coefficient

$$B_2 = \text{Coeff } q^N y^{\tilde{J}} \text{ in } \frac{\partial^2}{\partial \beta^2} f(\beta, q, y)|_{\beta=0} \quad (2.74)$$

In  $\mathcal{H}_k$  there is an orbifold action by  $\mathbb{Z}_k$ . The trace in the untwisted sector gives

$$\text{Tr}_{\mathcal{H}_k} (-1)^{2J-2h} e^{\beta 2h} q^{L_0} y^{2J} = \sum c(km, \tilde{l}, l) q^m y^l e^{\beta \tilde{l}}. \quad (2.75)$$

after projecting out states which are not invariant under  $\mathbb{Z}_k$ . We are now in good position to apply the theorem (2.15) in [63] which tells how to trace over  $S^r(\mathcal{H}_k)$ ,

$$\sum_{N \geq 0} p^N \text{Tr}_{S^N(\mathcal{H}_k)} (-1)^{2J-2h} e^{\beta 2h} q^{L_0} y^J = \prod_{n, \tilde{l}, l} \frac{1}{(1 - pq^n y^l e^{\beta \tilde{l}})^{c(kn, \tilde{l}, l)}}. \quad (2.76)$$

The function  $f(\beta, q, y)$  is given by the coefficient of  $p^r$  in the expression above. Even if this seems a hard task it is easier to perform first the step (2.74) and then extract the  $p^r$  coefficient, that is,

$$\text{Coeff } p^r \text{ off } \frac{\partial^2}{\partial \beta^2} f(\beta, q, y)|_{\beta=0} = \sum_{n, \tilde{l}, l} \tilde{l}^2 c(nk, \tilde{l}, l) r q^{nr} y^{lr} \quad (2.77)$$

where we have used the conditions

$$\sum_{\tilde{l}} c(n, \tilde{l}, l) = 0 \quad (2.78)$$

$$\sum_{\tilde{l}} \tilde{l} c(n, \tilde{l}, l) = 0 \quad (2.79)$$

that follow from the presence of two complex fermion zero modes in  $\mathcal{H}_1$ .

The coefficient of  $q^N y^{\tilde{J}}$  in (2.77) gives the index  $B_2$

$$B_2 = \sum_{s|I} s \sum_{\tilde{l}} \tilde{l}^2 c(NI/s^2, \tilde{J}/s, \tilde{l}) \quad (2.80)$$

where  $\hat{c}(n, l) = \sum_{\tilde{l}} \tilde{l}^2 c(n, l, \tilde{l})$  as in (2.69).

In the limit of large charges the term  $s = 1$  in (2.68) gives the leading contribution

$$B_2(Q_1 Q_5, n, l \gg 1) \approx e^{2\pi \sqrt{Q_1 Q_5 n I - l^2 I^2}} \quad (2.81)$$

which is in agreement with the Beckenstein-Hawking entropy of the 5d black hole in ALE space [46].

In the four dimensional case we have to consider in addition the closed string excitations of multi KK monopole.

We separate the problem in two pieces. First we try to argue that the Hilbert space of KK-P states is the hilbert space of multiply wound strings using IIB-Heterotic duality in four dimensions. Then using the 4d-5d lift together with fermion zero modes and duality invariance, we suggest that the Hilbert space of non-primitive dyons, denoted  $\mathcal{H}_I$ , is of the form

$$\mathcal{H}_I = \sum_{\sum k N_k = I} \prod_k S^{N_k}(\mathcal{H}_k) \quad (2.82)$$

where  $\mathcal{H}$  is the Hilbert space of a primitive dyon much like in the five dimensional case.

As mentioned at the beginning of this section, the study of bound states of multi KK monopoles by quantizing the moduli space is a very difficult problem. To circumvent this we map KK-P states in IIB to heterotic perturbative momentum winding states after performing T and six dimensional string-string dualities.

We want to study bound states of momentum  $nI$  and winding  $I$  at weak coupling. Because both charges have a common factor  $I$  the multiply wound string can split without breaking supersymmetry when

$$M(nI, I) = \sum_{\sum r k_r = I} r M(n k_r, k_r) \quad (2.83)$$

where  $M(n, w)$  is the BPS mass of a string with momentum  $n$  and winding  $w$ . This suggests that the Hilbert space of a multiply wound string is graded by partitions of  $I$

$$\mathcal{H}_I^{KK-P} = \sum_{\sum r k_r = I} \prod_r S^r(\mathcal{H}_{k_r}) \quad (2.84)$$

where  $\mathcal{H}_{k_r}$  corresponds to a  $k_r$  multiply wound string. Again the presence of fermion zero modes forces the partitions to obey  $r k_r = I$ . Only in this case the index is non vanishing. A state in  $\mathcal{H}_{k_r}$  carries momentum  $nI/r$  and winding  $I/r$ .

Naively, we would tensor (2.84) with (2.70) to obtain the full hilbert space of non-primitive dyons

$$\mathcal{H}_{full} = \sum_{r|I} S^r(\mathcal{H}_{I/r}^1) \otimes \sum_{k|I} S^k(\mathcal{H}_{I/k}^2) \quad (2.85)$$

but tracing the index  $B_6$  (2.3) over this space would give a non-duality invariant answer. Instead we propose that KK-P states should be symmetrized along with D1-D5 states in such a way that

$$\mathcal{H}_{full} = \mathcal{H}^0 \otimes \sum_{r|I} S^r(\mathcal{H}_{I/r}^1 \otimes \mathcal{H}_{I/r}^2) \quad (2.86)$$

A state in  $S^r(\mathcal{H}_{I/r}^1 \otimes \mathcal{H}_{I/r}^2)$  is of the form  $\sum_{\sigma} \epsilon(\sigma) \prod |n_i^1, h_i^1\rangle \otimes |n_i^2, J_i, h_i^2\rangle$  with total momentum  $\sum n_i^1 + n_i^2 = N$ , angular momentum  $\sum 2J_i = \tilde{J}$  and total helicity  $\sum 2h_i^1 + h_i^2$ . A state in  $\mathcal{H}^0$  is made up of four fermionic zero mode states  $\prod_i^4 \otimes |h_i = \pm 1/4\rangle$  while a state in the symmetric product carries only two fermion zero modes with  $h_i^2 = \pm 1/4$ . This distinction is based on the fact that we shouldn't trace over the center of mass degrees of freedom of the black hole. Since the black hole is free to move in the transverse  $\mathbb{R}^3$ , it will have bosonic zero modes along with the corresponding fermionic partners.

Such construction gives a duality invariant answer. Consider first the trace over the fermionic zero mode states in  $\mathcal{H}^0$ ,

$$B_6 = -\frac{1}{6!} \text{Tr}_{\mathcal{H}^0 \otimes \mathcal{H}^{II}} (-1)^{2J-2(h^0+h^I)} (2h^0 + 2h^I)^6 \quad (2.87)$$

$$= -\frac{1}{6!} \text{Tr}_{\mathcal{H}^0 \otimes \mathcal{H}^{II}} (-1)^{2J-2(h^0+h^I)} \frac{6!}{4!2!} (2h^0)^4 (2h^I)^2 \quad (2.88)$$

$$= -\frac{1}{2} \text{Tr}_{\mathcal{H}^{II}} (-1)^{2J-2h} (2h)^2 \quad (2.89)$$

where we have denoted  $S^r(\mathcal{H}_{I/r}^1 \otimes \mathcal{H}_{I/r}^2)$  by  $\mathcal{H}^{II}$ . The final trace has the form of the helicity trace  $B_2$ .

Following the same steps used to compute  $B_2$  in the five dimensional case, we arrive at the final answer

$$B_6 = \sum_{s|I} s c(Q_1 Q_5, nI^2/s^2, JI/s) \quad (2.90)$$

with the coefficient  $c(k, l, j)$  extracted from the primitive answer.

Further analysis of wall crossing phenomena in  $\mathcal{N} = 4$  string theory based on multi center black hole splitting [41] suggests that only two fermion zero modes have to be symmetrized. If we were to symmetrize  $n$  complex fermion zero modes we would get a factor of  $s^{n-1}$  in (2.90). A number  $n$  different from two would contradict the results from wall crossing and field theory dyon degeneracy. We don't have a physical explanation of why this should be the case.

In terms of the effective string this hilbert space corresponds to a sigma model with target space

$$\text{Sym}^I(\sigma_{D1-D5-KK}) \quad (2.91)$$

where  $\sigma_{D1-D5-KK}$  is the sigma model we described in section §2.3.1 for the primitive case.

## 2.5 A Perturbative test of the dyon counting formula

The non-primitive answer (2.53) is consistent with many physical tests. It reproduces the Beckenstein-Hawking entropy for large charges, it correctly reproduces wall crossing phenomena and the degeneracy of  $SU(N)$  field theory dyons are correctly captured for small charges.

Here we devise another microscopic test for the counting formula [34]. The strategy is to identify some states which are non-perturbative in one frame but are perturbative in another. This strategy has been used before in the case of half BPS states in various dualities. However in  $\mathcal{N} = 4$  gauge theory, the quarter BPS states are necessarily non-perturbative and cannot be mapped to any perturbative state. The reason is that the only perturbative states in the gauge theory are the gauge bosons which are half BPS. Interestingly this is not the case in string theory. A particular set of quarter BPS states in  $\mathcal{N} = 4$  can be mapped to perturbative states.

Consider the charge configuration

$$\Gamma = \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} 0 & n & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}_H.$$

It is easy to see that for these states the continuous T-duality invariants all vanish,  $Q^2 = P^2 = Q.P = 0$ . Nevertheless the invariant  $I$  is non-trivial,  $I = n$ .

Under six-dimensional string-string duality, the heterotic NS5-brane is mapped to type IIA fundamental string, and the momenta are mapped to momenta. Thus, in the Type-II frame, our state corresponds to a perturbative type II fundamental string with winding number one with  $n$  units of momentum along the  $S$  circle.

The non-perturbative counting through formula (2.53) gives

$$B_6(Q^2 = 0, P^2 = 0, Q.P = 0) = \sum_{s|I} s d_1(0, 0, 0) \quad (2.92)$$

The particular fourier coefficient  $d_1(0, 0, 0)$  of  $\Phi_{10}^{-1}$  vanishes. Consequently the index  $B_6$  vanishes for this charge configuration independently of the invariant  $I$ ,

$$B_6(Q^2 = 0, P^2 = 0, Q.P = 0, I) = 0 \quad (2.93)$$

This charge configuration maps to a fundamental string with unit winding and  $n$  units of momentum along  $S^1$  in type IIA on  $K3 \times S^1 \tilde{S}^1$ . To identify the spectrum we can use perturbative string theory.

For this propose we use light-cone gauge in Green-Schwarz formalism of string theory. The world-sheet theory thus have a target manifold  $\mathbb{R}^2 \times T^2 \times T^4/\mathbb{Z}_2$ , where we denote  $T^4/\mathbb{Z}_2$  the orbifold limit of  $K3$ .

We first compute the partition function

$$Z(q, \bar{q}, y) = \text{Tr}(-1)^F q^{L_0} \bar{q}^{\bar{L}_0} y^J \quad (2.94)$$

where  $J$  is the  $U(1)$  spin generator in the non-compact directions. The index  $B_6$  is extracted differentiating  $Z(q, \bar{q}, y)$  with respect to  $y$  six times and then setting  $y = 1$ . Under this process only quarter BPS states should be captured.

The partition function is

$$Z(q, \bar{q}, y) = (y^{1/2} - y^{-1/2})^4 \prod_{n \geq 1, j = \pm 1} \frac{(1 - \bar{q}^n y^j)^2 (1 - q^n y^j)^2}{(1 - \bar{q}^n)(1 - q^n)^2 (1 - \bar{q}^n y^{2j})^2 (1 - q^n y^{2j})^2} \times \text{Tr}_{K3}(q, \bar{q}, y) \quad (2.95)$$

with  $\text{Tr}_{K3}(q, \bar{q}, y)$  defined via

$$\text{Tr}_{K3}(q, \bar{q}, y) = 8 \left[ \frac{\vartheta_2(\tau, v)^2 \vartheta_2(\bar{\tau}, v)^2}{\vartheta_2(\tau, 0)^2 \vartheta_2(\bar{\tau}, 0)^2} + \frac{\vartheta_3(\tau, v)^2 \vartheta_3(\bar{\tau}, v)^2}{\vartheta_3(\tau, 0)^2 \vartheta_3(\bar{\tau}, 0)^2} + \frac{\vartheta_4(\tau, v)^2 \vartheta_4(\bar{\tau}, v)^2}{\vartheta_4(\tau, 0)^2 \vartheta_4(\bar{\tau}, 0)^2} \right] \quad (2.96)$$

$$q = e^{2\pi i \tau}, \bar{q} = e^{2\pi i \bar{\tau}}, y = e^{2\pi i v}$$

We refer the reader to [34] for further details. The partition function (2.95) contains already a factor of  $(y^{1/2} - y^{-1/2})^4$  due to the presence of eight real fermion zero modes in the Green-Schwarz formalism. For simplicity we remove first this factor and then differentiate twice. Imposing the level matching condition  $L_0 - \bar{L}_0 = I$  on the left-moving BPS states we get

$$d(I) = 16 \left[ \sum_{s|I} s(3 + (-1)^{s+1}) - 4 \sum_{(2s+1)|I} \frac{I}{2s+1} \right] \quad (2.97)$$

which can be shown to vanish identically for any value of  $I$ . This is in perfect agreement with the result (2.93). The same result was found in [64] as a consequence of a theta identity.

Note that for  $n = 0$ , we actually have a half-BPS state which is dual to a heterotic perturbative state. Since it breaks eight supersymmetries it carries four complex fermion zero modes so we need to differentiate the partition function four times. One correctly obtains a non-zero degeneracy which moreover equals 24 consistent with the heterotic counting [65].

### 3. $AdS_2/CFT_1$ correspondence and Quantum Entropy

For a large class of supersymmetric black holes in string theory, the Beckenstein-Hawking entropy  $S_{BH}$  finds perfect agreement with the logarithm of a microscopic index  $B_{micro}$  in the limit of large charges,

$$S_{BH} = \ln B_{micro} \quad (3.1)$$

In this limit both the computations simplify. On the microscopic side we can use an asymptotic expansion of the index instead of computing it exactly, while on the gravity side, due to a large horizon, we can neglect higher derivative corrections and work only with two derivative terms in the full string action. It is of great interest to know if the agreement holds for finite charges and if that is the case how to compute finite charge corrections.

As explained in the previous section, it is possible to compute exactly the index  $B_{micro}$  even for small charges for a large class of supersymmetric black holes in  $\mathcal{N} = 4$  string theory [26, 30, 29, 66]. The use of an index instead of degeneracy is of great advantage. Because

it captures only the BPS states, we can compare a microscopic computation with another performed when the black hole exists.

From the gravity side, the entropy function [67, 7] is a powerful way to compute the black hole entropy. Based on Wald’s formalism it gives a useful prescription to compute finite charge corrections to the entropy. Instead of computing a complicated integral over the horizon, as demanded by Wald’s formalism, it instructs us to minimize the lagrangian computed on the horizon solution. The entropy is then equal to the minima of that function which reduces the problem to solve some algebraic equations [68].

Introduction of higher derivative/loop string corrections through the entropy function in a consistent way is problematic. In the full quantum theory we have to integrate over massless fields and as consequence non-local terms in the action can be generated. This is problematic since Wald’s method requires a local, gauge and diffeomorphic invariant action.

To avoid this problem, Sen proposes a new framework based on  $AdS_2/CFT_1$  correspondence [9, 8]. The idea is to take the minimization process in the “classical” entropy function seriously by considering a path integral formulation. The Wald’s entropy then corresponds to the classical saddle point approximation of this path integral. Indeed, via holographic correspondence, we can relate the degeneracy of states in the  $CFT_1$  to a  $AdS_2$  path integral of string theory with a Wilson line insertion at the boundary. This is very powerful in the sense that it gives a consistent framework to compute both perturbative and non-perturbative charge corrections to the entropy.

Additional care of the microscopic index is required if we want to match the microscopic answer with the horizon  $AdS_2$  partition function. The reason is the following. The space-time index  $B_6$  captures all the degrees of freedom coming both from the horizon and any other contribution sitting between the horizon and asymptotic infinity. We call these additional degrees of freedom hair modes [69, 70]. Basically they correspond to deformations of the black hole solution with support outside of the horizon. On the microscopic side, they can correspond to the center of mass degrees of freedom of some brane system, for example.

This section is organized as follows. In section §3.1 we develop the concept of Sen’s quantum entropy function based on the  $AdS_2/CFT_1$  correspondence. In section §3.2 we explain the relation between index and degeneracy.

### 3.1 Quantum entropy

The entropy function which is based on the Wald formalism [4, 71, 5, 72] relates the value of the classical string theory lagrangian, calculated in the near horizon geometry, to the black hole entropy [67, 7, 68]. So we expect the entropy to depend only on the horizon data. For extremal black holes this assumption is even better because the horizon region is separated from asymptotic infinity by an infinite throat. In this case we can expect the entropy to depend on the horizon data not just classically, via Wald’s formalism, but also quantum mechanically. This is one of the pillars of Sen’s quantum entropy function that we develop in the following.

The quantum entropy formalism [9, 8, 73] uses  $AdS_2/CFT_1$  correspondence to give a quantum formulation of the black hole entropy. It states that

$$d_{hor}(q_I) = \left\langle e^{-iq_I \oint A^I} \right\rangle_{AdS_2}^{finite} \quad (3.2)$$

where  $d_{hor}$  is the degeneracy associated with the horizon of the black hole. In other words, the degeneracy  $d_{hor}(q)$  equals the expectation value of a Wilson line inserted at the boundary of euclidean  $AdS_2$ . The black hole entropy  $S_{BH}$  at the quantum level is given by the logarithm of  $d_{hor}$ . The symbol  $\langle \rangle_{AdS_2}$  in (3.2) denotes that we perform a path integral weighted by  $e^{-A}$  where  $A$  is the Euclidean string action.

We start by writing the near horizon field content of an extremal black hole after performing the Wick rotation  $t = -i\theta$ ,

$$ds^2 = v \left[ (r^2 - 1)d\theta^2 + \frac{dr^2}{r^2 - 1} \right], \quad \phi = \phi^*, \quad F_{r\theta}^I = -ie^I \quad (3.3)$$

where  $\phi^*$  are the attractor values of the scalars and  $F_{r\theta}^I$  are the  $U(1)$  gauge field strengths. The corresponding gauge field is  $A_\theta^I = -ie^I(r - 1)$  in the gauge  $A_r = 0$ . Since the thermal circle  $\theta$  is contractable in the  $AdS_2$  geometry this forces the gauge field to vanish for  $r = 0$ , otherwise it will be singular. The boundary stays at  $r = \infty$ .

Quantum mechanically, the  $AdS_2$  functional integral is defined by summing over all field configurations which asymptote to these attractor values with the fall-off conditions [8, 9, 74]

$$ds^2 = v \left[ (r^2 + \mathcal{O}(1)) d\theta^2 + \frac{dr^2}{r^2 + \mathcal{O}(1)} \right]. \quad (3.4)$$

$$\phi^I = \phi_*^I + \mathcal{O}(1/r), \quad A^I = -ie^I(r - \mathcal{O}(1))d\theta. \quad (3.5)$$

All massive fields asymptote to zero because of their mass term.

The path integral suffers from IR divergences due to the infinite volume of  $AdS_2$ . Therefore we introduce a cutoff at  $r = r_0$ . In an expansion in the cutoff parameter, the regulated amplitude then has the form

$$\langle \dots \rangle_{AdS_2} = e^{r_0 A + B + \mathcal{O}(r_0^{-1})} \quad (3.6)$$

The prescription used to remove the IR infinities is to keep only the finite part  $e^B$ . Technically we introduce a boundary counter term  $S_{bdy}$  to remove the contribution  $r_0 A$  and then take the limit  $r_0 \rightarrow \infty$ . The reader could have wondered why there aren't  $\ln(r_0)$  contributions. This has to do with the fact that in a perturbative expansion around the attractor background, say  $\phi = \phi^* + \delta\phi$ , with  $\phi = \mathcal{O}(1/r)$ , terms linear in  $1/r$  vanish via the equations of motion. So no logarithmic term is generated.

In defining the path integral on  $AdS_2$  we need to specify boundary conditions. Usual rules of  $AdS/CFT$  correspondence [36] instruct us to fix the non-normalizable modes and integrate over the normalizable ones. Special care is needed in the two dimensional case of Anti-de Sitter space. In this case the gauge field  $A$  has two solutions to the linearised Maxwell equations:  $A_\theta = c + ar$ ,

in the gauge  $A_r = 0$ . In contrast with higher dimensional cases, here the electric field mode is the dominant one so we should fix it. This is equivalent to working in the microcanonical ensemble where we fix the charges instead of the chemical potentials. The insertion of the Wilson line has the precise effect of rendering the equations of motion valid near the boundary [73]. Consider a small variation  $\delta A$  of the gauge field and look for the linearised equations of motion

$$\begin{aligned} \lim_{r \rightarrow \infty} -iq_I \int d\theta \delta A_\theta^I - \frac{\delta S}{\delta A_\theta^I} \delta A_\theta^I &= 0 \\ \lim_{r \rightarrow \infty} -iq_I \int d\theta \delta A_\theta^I - \int dr d\theta \frac{\delta \mathcal{L}}{\delta F_{r\theta}^I} \delta F_{r\theta}^I &= 0 \\ \lim_{r \rightarrow \infty} \left\{ -iq_I \int d\theta \delta A_\theta^I - \int d\theta \frac{\delta \mathcal{L}}{\delta F_{r\theta}^I} \delta A_\theta^I \right\} + \text{E.O.M.} &= 0 \end{aligned} \quad (3.7)$$

Since  $F_{r\theta}$  is non-zero at the boundary there is a non-trivial contribution to the linearised equations of motion at the boundary. If we hadn't introduced the Wilson loop, then the linearised equation (3.7) wouldn't be obeyed.

The other fields are fixed in the standard manner. For the metric field

$$ds^2 = v \left[ (r^2 - \mathcal{O}(1)) d\theta^2 + \frac{dr^2}{r^2 - \mathcal{O}(1)} \right] \quad (3.8)$$

we allow the constant mode denoted by  $\mathcal{O}(1)$  to fluctuate. On the other end for the scalar fields we fix the constant mode to the attractor value.

On the CFT side we should be computing

$$\text{Tre}^{-\beta H} \quad (3.9)$$

where  $H$  is the hamiltonian that generates translations on the boundary and the parameter  $\beta = T^{-1}$  is the inverse of the temperature. Holography relates the radius of the thermal circle to the temperature as  $\beta = 2\pi r_0$ . If the spectrum of  $H$  has a mass gap then only the ground states contribute to the trace when we take the zero temperature limit, or  $r_0 \rightarrow \infty$ . This dual quantum mechanics should be understood as the infrared limit of the quantum mechanics describing the black hole after removing the hair modes and it has the particular and interesting property that its hamiltonian is zero. In other words the quantum entropy function counts the number of ground states of the  $CFT_1$  in a particular charge and angular momentum<sup>5</sup> sector.

The initial intuition was that the quantum entropy function should reduce to the Wald entropy in limit of large charges. Say that we consider a perturbative solution around the attractor vacuum. In the limit of large charges we can carry out a saddle point approximation

$$\langle e^{-iq_I \oint A^I} \rangle \approx e^{-2\pi q_I e^I (r-1) + \int v dr d\theta \mathcal{L}(\phi^*, e^J) + \mathcal{O}(q^{-1})} \quad (3.10)$$

$$\approx e^{-2\pi q_I e^I (r_0-1) + 2\pi v (r_0-1) \mathcal{L}(\phi^*, e^J)} \quad (3.11)$$

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<sup>5</sup>If the black hole carries angular momentum then the horizon has isometry  $SO(2, 1) \times U(1)$  and the angular momentum is seen as a charge from the two dimensional point of view



where  $\mathcal{L}(\phi^*, e^J)$  is the lagrangian which is scalar, and  $\phi^*$  denotes the attractor values of the moduli. The gauge field chemical potentials are fixed to constant values such that the gauge field is regular at the origin  $r = 1$  of  $AdS_2$ , that is,  $A_\theta(r = 1) = 0$ . Following the prescription (3.6) we keep only the constant term, that is

$$\langle e^{-iq_I \oint A^I} \rangle^{finite} \approx e^{2\pi q_I e^I - 2\pi v \mathcal{L}(\phi^*, e^J)}. \quad (3.12)$$

This is exactly the exponential of the entropy function computed at the attractor value of the fields. Note that we have used the action principle  $e^{S_{eucl}}$  and not  $e^{-S_{eucl}}$  as is often used. This is related to the euclidean continuation chosen. Under  $t = -i\theta$ ,  $e^{iS}$  becomes  $e^{S_{eucl}}$  following [9, 73]. This is important because it is the renormalized action that should damp the path integral, which is what happens in this case. From now on we define the renormalized action  $S_{ren}$  to include also the Wilson loop contribution

$$\mathcal{S}_{ren} := \mathcal{S}_{bulk} + \mathcal{S}_{bdry} - i q_i \int_0^{2\pi} A_\theta^i d\theta \quad (3.13)$$

where  $\mathcal{S}_{bdry}$  is a boundary counterterm which renders the action IR finite. In addition we define the expectation value of the Wilson Loop in  $AdS_2$  by

$$W(q, p) = \left\langle \exp \left[ -i q_i \int_0^{2\pi} A_\theta^i d\theta \right] \right\rangle_{AdS_2}^{finite}. \quad (3.14)$$

At the classical attractor saddle point,

$$W(q, p) \sim \exp[2\pi(q_i e^i - v \mathcal{L})] \equiv \exp[S_{Wald}(q, p)] \quad , \quad (3.15)$$

This is a very powerful method to compute finite charge corrections to the Wald entropy. An immediate application consists in extending the perturbative analysis beyond the classical approximation. This is the work of [75, 76, 77] where the authors compute logarithmic corrections coming from a one-loop determinant in various supersymmetric theories.

Another non-trivial aspect of this formalism is that it allows for the contribution of additional subleading  $AdS_2$  orbifolds [12, 73]. They seem to play a non-trivial role in explaining non-perturbative contributions to the entropy as expected from microscopics [54].

### 3.2 Index versus degeneracy

The idea that only the horizon degrees of freedom are relevant for the black hole entropy leads automatically to the conclusion that two black holes with same near horizon geometry must have the same entropy. Nevertheless, the same is not true for the microscopic index. Two black holes can have the same entropy but carry a different index. This suggests that the horizon degeneracy should be combined with an exterior contribution to account for the difference.

This idea is consistent with the fact that even if the black hole entropy doesn't depend on the asymptotic values of the moduli, the index computed from microscopics can jump once

we vary the asymptotic values of the moduli [40]. It is known that for given a set of charges one can have single center as multi center black hole solutions [39, 43, 44]. The microscopic index doesn't know whether a state corresponds to a single or multi center solution, so it should include in the counting all these possibilities. Now, a multi center solution can cease to exist once we cross a wall of marginal stability which causes the index to jump [43, 56]. This also means that the entropy of a single center black hole can never jump. This suggests that string theory in the near horizon should capture only the degrees of freedom of a single black hole.

This is all suggestive to rewrite the index as [69, 70]

$$B_{micro}(q) = \sum_n \prod_{\substack{i=1 \\ \sum_i q_1^i + q_2^i = q}}^n d_{hor}(q_1^i) d_{hair}(q_2^i) \quad (3.16)$$

where the  $n^{th}$  term corresponds to the contribution of a  $n$ -centered black hole configuration,  $d_{hor}$  is the degeneracy associated with the horizon degrees of freedom and  $d_{hair}$  corresponds to an additional contribution coming from modes exterior to the horizon that we generically call hair modes. By setting the asymptotic values of the moduli to their attractor values we can guaranty that only the single center solution will contribute.

Consider the following puzzle which arises when trying to compare the index with the black hole entropy. Take the BMPV black hole [53]. Microscopically it corresponds to a system of D1-D5 branes wrapping  $K3 \times S^1$  and carrying momentum along  $S^1$ . It can also carry angular momentum without breaking supersymmetry. Now consider the D1-D5-KK system discussed in section §2. The 4D-5D lift relates these two configurations. If we put the BMPV black hole at the tip of the Taub-Nut it becomes the D1-D5-KK black hole. This means that if we zoom close to the origin of the Taub-Nut, where the space looks flat, both black hole solutions will look the same. As a matter of fact, this is also equivalent to the near horizon limit. This means that both the BMPV and D1-D5-KK black holes have the same near horizon geometry and hence the same black hole entropy. Nevertheless, the index differs substantially from one configuration to the other [26, 27]. We expect that after removing the hair contribution both degeneracies will agree.

The strategy is to study normalizable deformations of the black hole solution and check whether they have or not support near the horizon. Those that vanish near the horizon correspond to hair modes. In [69] the authors analysed the deformations using linearised equations of motion.

The first basic conclusion is that all the fermion zero modes are part of the hair degrees of freedom. This not surprising since the solution outside the horizon breaks supersymmetry and therefore the goldstino modes must have support outside of the horizon. Additionally, they found that for the BMPV black hole the center of mass modes of the D1-D5 system are part of the hair degrees of freedom.

For the D1-D5-KK black hole they have also found that the center of mass degrees of freedom of the D1-D5 moving in the transverse Taub-Nut space are also part of the hair degrees of

freedom. The additional contribution coming from closed string excitations of the KK solution is also part of the hair modes.

The conclusion is that after removing the hair contribution the microscopic horizon partitions for both of the black holes agree,

$$Z_{5D}^{hor} = Z_{4D}^{hor} \quad (3.17)$$

They also make the important observation that these new partition functions are free from poles which could induce jumps in the index. This is important if we want the black hole entropy to not have moduli dependence.

A more refined approach in [70] using non-linear equations arrives at the same conclusion (3.17) but with some important differences. The analysis shows that the bosonic deformations corresponding to the center of mass degrees of freedom of the brane system have curvature singularities at future horizon. Hence they should be include as horizon modes.

In the following we show why, for a class of supersymmetric black holes, the index matches with the black hole degeneracy in the large charge limit.

In section §2 we defined the helicity trace index  $B_6$ , suitable to capture the spectrum of quarter BPS dyons in  $\mathcal{N} = 4$  theory. It was defined as

$$B_6 = -\frac{1}{6!} \text{Tr}(-1)^{2h} (2h)^6 \quad (3.18)$$

with  $h$  the helicity quantum number in four dimensions. We justified the inclusion of six powers of  $(2h)$  in the trace to remove the contribution of six complex fermion zero modes coming from the breaking of twelve supersymmetries, and this way rendering the index non vanishing.

In the same spirit we can rewrite the index  $B_6$  in terms of horizon and hair contributions as

$$B_6 = -\frac{1}{6!} \text{Tr}(-1)^{2h_{hor}+2h_{hair}} (2h_{hor} + 2h_{hair})^6 \quad (3.19)$$

$$= -\frac{1}{6!} \text{Tr}(-1)^{2h_{hor}} \text{Tr}(-1)^{2h_{hair}} (2h_{hair})^6 \quad (3.20)$$

$$= \sum_{q+\tilde{q}=Q} B_{hor}(q) B_{6\,hair}(\tilde{q}) \quad (3.21)$$

where we used the fact that only the term  $(2h_{hair})^6$  survives in the taylor expansion of  $(2h_{hor} + 2h_{hair})^6$  since the fermion zero modes are part of the hair degrees of freedom. The index  $B_{hor}$  for the horizon degrees of freedom is given by the Witten index

$$B_{hor} = \text{Tr}(-1)^{2h_{hor}} \quad (3.22)$$

and the index  $B_{6\,hair}$  for the hair modes was defined as

$$B_{6\,hair} = -\frac{1}{6!} \text{Tr}(-1)^{2h_{hair}} (2h_{hair})^6. \quad (3.23)$$

If all the hair degrees of freedom are fermionic zero modes than  $B_{6\text{hair}} = -1$  implying  $B_6 = \text{Tr}(-1)^F$ .

For an extremal spherically symmetric black hole all states carry  $h_{\text{hor}} = 0$  implying the equality  $\text{Tr}(-1)^{2h_{\text{hor}}} = \text{Tr}(1)$  [73]. This shows that index equals degeneracy for the horizon. Moreover, for a black hole that preserves at least four supercharges, closure of the supersymmetry algebra implies spherical symmetry. In other words, a  $SU(2)$  subalgebra is necessary for the susy algebra to close and this factor can be identified with the rotation symmetry of the near horizon geometry. The index  $B_{\text{hor}}$  can be computed using the quantum entropy function [9, 73] which in the limit of large charges reduces to the exponential of the wald entropy. This, together with the fact that the hair contribution to the index is usually negligible compared to the wald entropy, explains why for large charges index equals degeneracy

$$B_6(Q, P \gg 1) \approx B_{\text{hor}}(Q, P \gg 1) \approx e^{S_{BH}(Q, P)}. \quad (3.24)$$

This also explains why for one-sixteenth BPS black holes in  $AdS_5$  no microscopic index seems to have the right asymptotics consistent with black hole entropy [78, 79, 80]. Since in this case the black holes preserves too little supersymmetry closure of the supersymmetry algebra does not imply rotational invariance.

## 4. Quantum black holes and Localization

The Bekenstein-Hawking entropy is in a sense a bit too universal in that it is always given by a quarter of the horizon area. This is a consequence that, for very large distances, only the Einstein-Hilbert term in the action contributes. Finite charge corrections, on the other hand, can arise after introduction of higher derivative terms which are different in different phases of the theory. This dependence on the phase can yield useful information about different aspects of the short-distance theory. In this section we are interested in computing finite charge corrections by explicitly evaluating the quantum entropy function for supersymmetric black holes in a broad class of phases of string theory, namely vacua with  $\mathcal{N} = 2$  supersymmetry in four dimensions [17].

In a theory with massless  $n_v + 1$  vector fields, a black hole is specified by a charge vector  $(q_I, p^I)$  with  $I = 0, \dots, n_v$ . We would like to develop methods to systematically compute the quantum entropy for arbitrary finite values of the charges. As explained in section §3 the quantum entropy function, via  $AdS_2/CFT_1$  correspondence, gives a consistent and powerful framework to compute perturbative and non-perturbative corrections.

Via  $AdS_2/CFT_1$  correspondence the microscopic degeneracy  $d(q, p)$  is identified with the expectation value of the Wilson loop that we denote by  $W(q, p)$ . Evaluating the formal expression for  $W(q, p)$  by doing the string field theory functional integral is of course highly nontrivial. To proceed further we imagine first integrating out the infinite tower of massive string modes and massive Kaluza-Klein modes to obtain a *local* Wilsonian effective action for the massless supergravity fields. To compute the exact quantum entropy, one has to then evaluate exactly this

functional integral of a finite number of massless fields with  $AdS_2$  boundary conditions using the full Wilsonian effective action keeping all higher derivative terms. This effective action can include in general not only perturbative corrections in  $\alpha'$  but also worldsheet instanton corrections. We can regard the ultraviolet finite string theory as providing a finite, supersymmetric, and consistent cutoff at the string scale. The functional integral with such a finite cut-off and a Wilsonian effective action containing all higher order terms is thus in principle free of ultraviolet divergences. This functional integral will be our starting point.

We are still left with the task of evaluating a complicated functional integral. The near horizon geometry preserves eight superconformal symmetries and moreover the action, measure, operator insertion, boundary conditions of the functional integral are all supersymmetric. This allows us to apply localization techniques [12] which simplifies the evaluation of the functional integral enormously. Localization requires identification of a fermionic symmetry of the theory that squares to a compact bosonic symmetry. Using this symmetry, one can then localize the functional integral onto the ‘localizing submanifold’ of bosonic field configurations invariant under the fermionic symmetry. We review the superconformal symmetries of the near horizon geometry and relevant aspects of localization in §4.1.

Since localization is employed at the level of the functional integral and not just at the level of a classical action, it is important to use an *off-shell* formulation of supergravity. Off-shell formulations of supergravity are in general notoriously involved. At present a complete formulation of off-shell supergravity coupled to both vector and hyper multiplets is not known. To implement localization in a concrete manner, we therefore first consider in §4.3 a simpler problem of computing this expectation value of the Wilson line in a truncated model of supergravity coupled only to vector multiplets with an action containing only F-terms which are chiral integrals over superspace. In particular we ignore possible D-terms and hyper multiplets, which are discussed later in §4.4. The action still contains an infinite number of higher derivative terms but all of F-type. We denote the corresponding functional integral for the expectation value of a Wilson line in this restricted theory on  $AdS_2$  by  $\widehat{W}(q, p)$ . Computation of  $\widehat{W}(q, p)$  is greatly simplified by the fact that, for vector multiplets in  $\mathcal{N} = 2$  supergravity, there exists an elegant off-shell formulation developed in [13, 15, 14], using the superconformal calculus. The spectrum consists of the Weyl multiplet that contains the graviton and the gravitini,  $n_v + 1$  vector multiplets, and one compensating multiplet that eliminates unwanted degrees of freedom. We review this formalism in §4.2.

The main result of this section concerns the localization of the functional integral for  $\widehat{W}(q, p)$  which is derived in §4.3.

The organization of this section is as follows. We start by reviewing the technique of localization and the superconformal symmetries of the near horizon geometry. In section §4.2 we review the superconformal construction of supergravity using F-terms. In section §4.3 we apply localization and determine  $\widehat{W}(q, p)$ . We end commenting on limitations of this approach and also on possible contributions from D-terms and hypermultiplets.

## 4.1 Superconformal symmetries and localization

We start with a brief review in §4.1.1 of the localization techniques [81, 82, 83, 84, 10, 85] to evaluate supersymmetric functional integrals. In §4.1.2 we review the superconformal symmetries of the attractor geometry and how localization can be applied in the present context.

### 4.1.1 A review of localization of supersymmetric functional integrals

Consider a supermanifold  $\mathcal{M}$  with an integration measure  $d\mu$ . Let  $Q$  be an odd (fermionic) vector field on this manifold that satisfies the following two requirements:

1.  $Q^2 = H$  for some compact bosonic vector field  $H$ ,
2. The measure is invariant under  $Q$ , in other words  $\text{div}_\mu Q = 0$ .

The divergence of the fermionic vector field is the natural generalization of ordinary divergence, which satisfies in particular<sup>6</sup>

$$\int_{\mathcal{M}} d\mu Q(f) = - \int_{\mathcal{M}} d\mu (\text{div}_\mu Q) f, \quad (4.1)$$

for any function  $f$ . Hence, the second property implies  $\int_{\mathcal{M}} d\mu Q(f) = 0$  for any  $f$ . We would like to evaluate an integral of some  $Q$ -invariant function  $h$  and a  $Q$ -invariant action  $S$

$$I := \int_{\mathcal{M}} d\mu h e^{-S}. \quad (4.2)$$

To evaluate this integral using localization, one first deforms the integral to

$$I(\lambda) = \int_{\mathcal{M}} d\mu h e^{-S - \lambda QV}, \quad (4.3)$$

where  $V$  is a fermionic,  $H$ -invariant function which means  $Q^2 V = 0$ , that is,  $QV$  is  $Q$ -exact. One has

$$\frac{d}{d\lambda} \int_{\mathcal{M}} d\mu h e^{-S - \lambda QV} = \int_{\mathcal{M}} d\mu h QV e^{-S - \lambda QV} = \int_{\mathcal{M}} d\mu Q(h e^{-S - \lambda QV}) = 0, \quad (4.4)$$

and hence  $I(\lambda)$  is independent of  $\lambda$ . This implies that one can perform the integral  $I(\lambda)$  for any value of  $\lambda$  and in particular for  $\lambda \rightarrow \infty$ . In this limit, the functional integral localizes onto the critical points of the functional  $S^Q := QV$  and the semiclassical approximation becomes exact. The localizing solutions in general have both bosonic and fermionic collective coordinates.

One can choose

$$V = (Q\Psi, \Psi) \quad (4.5)$$

where  $\Psi$  are the fermionic coordinates with some positive definite inner product defined on the fermions. In this case, the bosonic part of  $S^Q$  can be written as a perfect square  $(Q\Psi, Q\Psi)$ , and

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<sup>6</sup>For a bosonic vector field  $V$  and for a measure determined by a metric  $g$ , this corresponds to the identity  $\int dx \sqrt{g} V^m \partial_m f = - \int dx \partial_m (\sqrt{g} V^m) f = - \int dx \sqrt{g} (\nabla_m V^m) f$  when the boundary contributions vanish.

hence critical points of  $S^Q$  are the same as the critical points of  $Q$ . This also implies that  $QV$  vanishes on the localizing manifold as required by the invariance of  $I(\lambda)$  (4.4). Let us denote this set of critical points of  $Q$  by  $\mathcal{M}_Q$ . The reasoning above shows that the integral over the supermanifold  $\mathcal{M}$  localizes to an integral over the submanifold  $\mathcal{M}_Q$ . In the large  $\lambda$  limit, the integration for directions transverse can be performed exactly in the saddle point evaluation. One is then left with an integral over the submanifold  $\mathcal{M}_Q$

$$I = \int_{\mathcal{M}_Q} d\mu_Q h e^{-S} \text{sdet}(D_2), \quad (4.6)$$

where  $d\mu_Q$  is a measure induced on the submanifold by the original measure and  $\text{sdet}(D_2)$  is the superdeterminant of transverse fluctuations. We denote  $D_2$  the operator of quadratic fluctuations of the  $QV$  action.

In our case in §4.3,  $\mathcal{M}$  is the field space of off-shell supergravity,  $\mathcal{S}$  is the off-shell supergravity action with appropriate boundary terms,  $h$  is the supersymmetric Wilson line,  $Q$  is a specific supercharge described in §A and §4.3, and  $\Psi$  are all fermionic fields of the theory. We will find that the submanifold  $\mathcal{M}_Q$  of localizing solutions is a family of nontrivial instantons as exact solutions to the equations of motion that follow from extremization of  $S^Q$  labeled by  $n_v + 1$  real parameters  $\{C^I; I = 0, \dots, n_v\}$ .

#### 4.1.2 Superconformal symmetries of the near horizon geometry

In higher dimensional cases we normally take the near horizon limit of an extremal, that is, zero temperature brane configuration. This limit allows us to focus on energy fluctuations of the brane system which are small to the asymptotic observer but sufficiently large compared to the temperature of the system. For the case of extremal black holes we proceed a bit differently [86]. Since the black hole quantum mechanics has a mass gap separating the ground state from the first excited state the only low energy excitations are zero energy excitations. This means that the usual near horizon limit of an extremal black hole is not a sensible limit. Instead we proceed by taking the near horizon and extremal limits at the same time [8].

The near-horizon geometry of a supersymmetric black hole in four dimensions is  $AdS_2 \times S^2$ . After Euclidean continuation, the metric is

$$ds^2 = v \left[ (r^2 - 1) d\theta^2 + \frac{dr^2}{r^2 - 1} \right] + v \left[ d\psi^2 + \sin^2(\psi) d\phi^2 \right]. \quad (4.7)$$

We have taken the radius  $v$  of the  $AdS_2$  factor to be the same as the radius of the  $S^2$  factor which is a consequence of supersymmetry. There are several other coordinates that are useful. Substituting  $r = \cosh(\eta)$ , the metric takes the form

$$ds^2 = v \left[ d\eta^2 + \sinh^2(\eta) d\theta^2 \right] + v \left[ d\psi^2 + \sin^2(\psi) d\phi^2 \right]. \quad (4.8)$$

One can also choose the stereographic coordinates

$$w = \tanh\left(\frac{\eta}{2}\right) e^{i\theta} := \rho e^{i\theta}, \quad z = \tan\left(\frac{\psi}{2}\right) e^{i\phi}, \quad (4.9)$$

in which the metric takes the form

$$ds^2 = v \frac{4dw d\bar{w}}{(1 - w\bar{w})^2} + v \frac{4dz d\bar{z}}{(1 + z\bar{z})^2}. \quad (4.10)$$

Note that the interval for the coordinates are  $1 \leq r < \infty$  and  $0 \leq \eta < \infty$ , and  $0 \leq \rho < 1$ . In the  $w$  coordinates, Euclidean  $AdS_2$  can be readily recognized as the Poincaré disk with  $\rho$  as the radial coordinate of the disk and a boundary at  $\rho = 1$ .

The Weyl tensor for the metric (4.8) is zero and hence this metric is conformally flat. For later use it will be useful to know this conformal transformation. To map we first map the Poincaré disk to the upper half plane by the transformation

$$u = x + iy, \quad u = i \frac{1 - iw}{1 + iw}. \quad (4.11)$$

The metric (4.7) in the new coordinates becomes

$$ds^2 = \frac{dx^2 + dy^2 + y^2 d\Omega_2^2}{y^2}, \quad (4.12)$$

with  $-\infty < x < +\infty$  and  $0 \leq y < \infty$ . From the above equation, we see that  $AdS_2 \times S^2$  is conformally flat. We also know that  $\mathbb{R}^4$  is conformal to  $S^4$  so it would be useful to compute the conformal factor relating  $AdS_2 \times S^2$  to  $S^4$ . In the  $(\eta, \theta)$  coordinates we have the following conformal rescaling

$$ds^2(AdS_2 \times S^2) = \cosh^2(\eta) ds^2(S^4). \quad (4.13)$$

Note that the conformal factor diverges at the boundary. Under a Weyl transformation

$$g_{\mu\nu} \rightarrow e^{2\Omega} g_{\mu\nu}, \quad (4.14)$$

a field with Weyl weight  $a$  transforms as

$$\Phi \rightarrow e^{-a\Omega} \Phi. \quad (4.15)$$

Hence, such a field in the conformal frame with  $AdS_2 \times S^2$  metric will be mapped to the field in the conformal frame with  $S^4$  metric by

$$\Phi_{AdS_2 \times S^2} = \frac{\Phi_{S^4}}{\cosh(\eta)^a}. \quad (4.16)$$

This transformation will be useful later in §4.3.

The superconformal symmetry of the near horizon geometry is the semidirect product  $SU(1,1|2) \ltimes SU(2)'$ . The invariant subgroup  $SU(1,1|2)$  will be of our main interest which contains the bosonic subgroup  $SU(1,1) \times SU(2)$ . The first factor can be identified with the conformal symmetry of  $AdS_2$  and is generated by  $\{L, L_\pm\}$ . The second factor can be identified with the rotational symmetry of  $S^2$  and is generated by  $\{J, J_\pm\}$ . The factor  $SU(2)'$  originates



from the R-symmetry of  $\mathcal{N} = 2$  supergravity in four dimensions. The odd elements of the superalgebra are the superconformal symmetries  $G_r^{ia}$ . The commutations relations are

$$[L, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = -2L, \quad (4.17)$$

$$[J, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = 2J, \quad (4.18)$$

$$[L, G_\pm^{ia}] = \pm \frac{1}{2} G_\pm^{ia}, \quad [L_\pm, G_\mp^{ia}] = -i G_\pm^{ia}, \quad (4.19)$$

$$[J, G_r^{i\pm}] = \pm \frac{1}{2} G_r^{i\pm}, \quad [J^\pm, G_r^{i\mp}] = G_r^{i\pm}, \quad (4.20)$$

$$\{G_+^{i\pm}, G_-^{j\pm}\} = \pm 4\epsilon^{ij} J^\pm, \quad \{G_\pm^{i+}, G_\pm^{j-}\} = \mp 4i\epsilon^{ij} L_\pm, \quad (4.21)$$

$$\{G_\pm^{i+}, G_\mp^{j-}\} = 4\epsilon^{ij} (L \mp J); \quad \epsilon^{+-} = -\epsilon^{-+} = 1. \quad (4.22)$$

Explicit expressions for the Killing spinors corresponding to these superconformal supersymmetries will be obtained in §A and will be required for localization in §4.3.

It is easy to see from the algebra that the generator  $Q = G_+^{++} + G_-^{--}$  squares to  $4(L - J)$ . Since  $L$  is the generator of rotations of the Poincaré disk and  $J$  is the generator of rotations of  $S^2$ , the square  $Q^2$  is the generator of a compact bosonic symmetry. This is the generator that we will use for localization.

## 4.2 Off-shell formulation of the theory

In this section, we review the off-shell formulation of supergravity due to [13, 15, 14]. This formalism has several attractive features.

- First, it allows the supersymmetry transformations to be realized in an off-shell manner which will be crucial for us to apply localization to the functional integral.
- Second, one can also include within the formalism a class of curvature squared corrections to the theory that are encoded in the Weyl multiplet. This has made it possible to study the higher derivative corrections to supersymmetric black holes using the full power of supersymmetry for solving BPS equations in the classical theory.
- Third, in the off-shell formalism, the supersymmetry transformations are specified once and for all and do not need to be modified as one modifies the action with higher derivative terms. This is analogous to the situation for diffeomorphisms where the transformation properties of the metric, for example, are specified once and for all and does not depend on the form the action. Since the localization action that we use is constructed using these supersymmetry transformations, the localizing solutions that we will obtain by minimizing this action will therefore be universal and not dependent on the form of the physical action. This is clearly greatly advantageous both at the technical and conceptual level.

In this section we rederive the classical properties of the black hole in this new language. This section is meant to set the stage and fix all the notations for the quantum calculation which

we discuss in §4.3. It will therefore be concise; a detailed account of the off-shell formalism can be found, for example, in the review [87].

We use the *conformal supergravity* approach to  $\mathcal{N} = 2$  off-shell supergravity in four dimensions developed using *superconformal multiplet calculus*. The main idea is to extend the symmetries of the  $\mathcal{N} = 2$  Poincaré supergravity to the  $\mathcal{N} = 2$  superconformal algebra. This bigger algebra has dilatations, special conformal transformations, conformal  $S$ -supersymmetries, and local  $SU(2) \times U(1)$  symmetries as extra symmetries compared to the Poincaré group<sup>7</sup>. The conformal supergravity is then constructed as a gauge theory of this extended symmetry group.

Upon gauge fixing the extra superconformal symmetries, one gets the Poincaré supergravity. In this sense, they are both gauge equivalent. However, the multiplet structure of the superconformal supergravity is smaller and simpler than the Poincaré theory. The form of the supersymmetry transformation rules is also simpler in the superconformal formalism, and one has a systematic way of deriving invariant Lagrangians. Following this approach, one gets an off-shell formulation of supergravity coupled to vector multiplets.

In §4.2.1, we first list the multiplets of the superconformal theory that will enter the theories we consider. In appendix §B, we summarize some relevant aspects of the superconformal multiplet calculus including the supersymmetry variations of the various multiplets listed below. In §4.2.2 we discuss the invariant action of our interest.

#### 4.2.1 Superconformal multiplets

Our *conventions* are as follows. In the Minkowski theory, all fermion fields below are represented by Majorana spinors. In the Euclidean theory, they will be symplectic-Weyl-Majorana [88]. Greek indices  $\mu, \nu, \dots$  indicate the curved spacetime, latin indices  $a, b, \dots$  indicate the flat tangent space indices, and  $i, j, \dots$  denote the  $SU(2)$  index. The  $SU(2)$  indices are raised and lowered by complex conjugation.  $A^- \equiv \varepsilon_{ij} A^{ij}$  for any  $SU(2)$  tensor  $A^{ij}$ . We will also use the superscript  $\pm$  to denote (anti) self-duality in spacetime, the conventions should be clear from the context. We use the covariant derivative  $D_a$ , which is defined to be covariant with respect to all the superconformal transformations as well as gauge fields of the theory if present. We also use the bosonic covariant derivative  $\nabla_a$  is defined to be covariant with respect to all the bosonic transformations and the gauge fields, except the special conformal transformation.

We now summarize the field content of various multiplets.

1. *Weyl multiplet*: This is the gravity multiplet which contains all gauge fields arising from gauging the full superconformal symmetries. The field content is:

$$\mathbf{w} = (e_\mu^a, w_\mu^{ab}, \psi_\mu^i, \phi_\mu^i, b_\mu, f_\mu^a, A_\mu, \mathcal{V}_{\mu j}^i, T_{ab}^{ij}, \chi^i, D) . \quad (4.23)$$

---

<sup>7</sup>Note that the extra superconformal symmetry of this formalism is a gauge symmetry, not to be confused with the physical superconformal algebra of the near-horizon geometry of extremal black holes discussed in §4.1.2 which is generated by the Killing vectors and Killing spinors of the background.

The fields  $(e_\mu^a, w_\mu^{ab})$  are the gauge fields for translations (vielbein) and Lorentz transformations;  $\psi_\mu^i, \phi_\mu^i$  are the gauge fields for Q-supersymmetries and the conformal S-supersymmetries;  $(b_\mu, f_\mu^a)$  are the gauge fields for dilatations and the special conformal transformations; and  $(\mathcal{V}_\mu^i, A_\mu)$  are the gauge fields for the  $SU(2)$  and  $U(1)$  R-symmetries. Imposition of the ‘conventional constraints’ determines  $w_\mu^{ab}, \phi_\mu^i, f_\mu^a$  in terms of other fields and one is left with  $24 + 24$  independent degrees of freedom. The  $SU(2)$  doublet of Majorana spinors  $\chi^i$ , the antisymmetric anti self-dual auxiliary field  $T_{ab}^{ij}$  and the real scalar field  $D$  are all auxiliary fields, some of which will play a non-trivial role later. This multiplet contains the gravitational degrees of freedom.

2. *Vector multiplet*: The field content is

$$\mathbf{X}^I = (X^I, \Omega_i^I, A_\mu^I, Y_{ij}^I) \quad (4.24)$$

with  $8 + 8$  degrees of freedom.  $X^I$  is a complex scalar, the gaugini  $\Omega_i^I$  are an  $SU(2)$  doublet of chiral fermions,  $A_\mu^I$  is a vector field, and  $Y_{ij}^I$  are an  $SU(2)$  triplet of auxiliary scalars. This multiplet contains the gauge field degrees of freedom.

3. *Chiral multiplet*: The field content is

$$\widehat{\mathbf{A}} = (\widehat{A}, \widehat{\Psi}_i, \widehat{B}_{ij}, \widehat{F}_{ab}^-, \widehat{\Lambda}_i, \widehat{C}) \quad (4.25)$$

with  $16 + 16$  components.  $\widehat{A}, \widehat{C}$  are complex scalars,  $\widehat{B}_{ij}$  is a complex  $SU(2)$  triplet,  $\widehat{F}_{ab}^-$  is an antiselfdual Lorentz tensor, and  $\widehat{\Psi}_i, \widehat{\Lambda}_i$  are  $SU(2)$  doublets of left-handed fermions. The action will also contain the conjugated right handed multiplet. One can impose a supersymmetric constraint on the chiral multiplet to get a reduced chiral multiplet with  $8 + 8$  degrees of freedom.

The covariant quantities of a vector multiplet are associated with a reduced chiral multiplet. The covariant quantities of the Weyl multiplet are also associated with a reduced chiral multiplet  $\mathbf{W}_{ab}^{ij}$ . Products of chiral multiplets are also chiral, and one thus gets a chiral multiplet  $\widehat{\mathbf{A}} = \mathbf{W}^2 = \varepsilon_{ik}\varepsilon_{jl}\mathbf{W}_{ab}^{ij}\mathbf{W}^{abkl}$ . The lowest component of  $\widehat{\mathbf{A}}$  is  $\widehat{A} = (T_{ab}^{ij}\varepsilon_{ij})^2$  and the highest component of  $\widehat{\mathbf{A}}$  contains terms quadratic and linear in the curvature. The problem of building Lagrangians with terms quadratic in the curvature thus reduces to the simpler problem of coupling the chiral multiplet  $\widehat{\mathbf{A}}$  to the superconformal theory.

4. *Compensating multiplet*: This multiplet will be used as a compensator to fix the extra gauge transformations. There are three types of compensators that have been used in the literature so far, a non-linear multiplet, a compensating hypermultiplet and a tensor multiplet. As an example, we discuss the non-linear multiplet [87, 89]. Other multiplets have their relative advantages, in particular the compensating hypermultiplet is used extensively for the treatment of higher derivative terms [90].

*Non-linear multiplet*:

$$(\Phi_\alpha^i, \lambda^i, M^{ij}, V_a) \quad (4.26)$$

where  $\lambda^i$  is a spinor  $SU(2)$  doublet,  $M^{ij}$  is a complex antisymmetric matrix of Lorentz scalars, and  $V_a$  is a real Lorentz vector.  $\Phi_\alpha^i$  is an  $SU(2)$  matrix of scalar fields with the  $\alpha$  index transforming in the fundamental of a rigid  $SU(2)$ , it describes three real scalars. Naively, the multiplet has  $9+8$  degrees of freedom, but there is a supersymmetric constraint on the vector  $V_a$  which reduces the degrees of freedom to  $8+8$ :

$$D^a V_a - 3D - \frac{1}{2} V^a V_a - \frac{1}{4} |M_{ij}|^2 + D^a \Phi_\alpha^i D_a \Phi_i^\alpha + \text{fermions} = 0 \quad (4.27)$$

#### 4.2.2 Superconformal action

The procedure to get invariant actions is as follows: one first finds an invariant Lagrangian for a chiral multiplet, this was solved in [16]. The second step is to write down a scalar function, the *prepotential*  $F(X^I)$  of the vector multiplets which is a meromorphic homogeneous function of weight 2. One then uses the chiral Lagrangian of the first step for the chiral multiplet  $\mathbf{F}$ . This gives the two derivative  $\mathcal{N} = 2$  Poincaré supergravity after gauge fixing. To include coupling to curvature square terms, one extends the function  $F$  to depend on the lowest component  $\hat{A}$  of the chiral multiplet  $\hat{\mathbf{A}} = \mathbf{W}^2$ .  $F(X^I, \hat{A})$  is holomorphic and homogeneous of degree two in all its variables. One then uses the chiral Lagrangian of the first step for the chiral multiplet  $\mathbf{F}$ .

We use the following notations. The prepotential which is a meromorphic function of its arguments obeys the homogeneity condition:

$$F(\lambda X, \lambda^2 \hat{A}) = \lambda^2 F(X, \hat{A}). \quad (4.28)$$

Its various derivatives are defined as:

$$F_I = \frac{\partial F}{\partial X^I}, \quad F_{\hat{A}} = \frac{\partial F}{\partial \hat{A}}, \quad F_{IJ} = \frac{\partial^2 F}{\partial X^I \partial X^J}, \quad F_{\hat{A}I} = \frac{\partial^2 F}{\partial X^I \partial \hat{A}}, \quad F_{\hat{A}\hat{A}} = \frac{\partial^2 F}{\partial \hat{A}^2}. \quad (4.29)$$

Following the above procedure, one gets a invariant action for  $I = 1, 2, \dots, N_V + 1$  vectors coupled to conformal supergravity. The bosonic part of the action is:

$$\begin{aligned} e^{-1} \mathcal{L} = & i \left[ \bar{F}_I X^I \left( \frac{1}{6} R - D \right) + \nabla_\mu F_I \nabla^\mu \bar{X}^I \right. \\ & + \frac{1}{4} F_{IJ} (F_{ab}^{-I} - \frac{1}{4} \bar{X}^I T_{ab}^{ij} \varepsilon_{ij}) (F^{-abJ} - \frac{1}{4} \bar{X}^J T_{ab}^{ij} \varepsilon_{ij}) \\ & - \frac{1}{8} F_I (F_{ab}^{+I} - \frac{1}{4} X^I T_{abij} \varepsilon^{ij}) T_{ab}^{ij} \varepsilon_{ij} - \frac{1}{8} F_{IJ} Y_{ij}^I Y^{Jij} - \frac{1}{32} F (T_{abij} \varepsilon^{ij})^2 \\ & + \frac{1}{2} F_{\hat{A}} \hat{C} - \frac{1}{8} F_{\hat{A}\hat{A}} (\varepsilon^{ik} \varepsilon^{jl} \hat{B}_{ij} \hat{B}_{kl} - 2 \hat{F}_{ab}^- \hat{F}_{ab}^-) + \frac{1}{2} \hat{F}^{-ab} F_{\hat{A}I} (F_{ab}^{-I} - \frac{1}{4} \bar{X}^I T_{ab}^{ij} \varepsilon_{ij}) \\ & \left. - \frac{1}{4} \hat{B}_{ij} F_{\hat{A}I} Y^{Iij} \right] + \text{h.c.} \end{aligned} \quad (4.30)$$

To get to the  $\mathcal{N} = 2$  Poincaré supergravity, one has to gauge fix the extra gauge transformations of the superconformal theory. To gauge fix the special conformal transformations, one sets the *K-gauge*:

$$b_\mu = 0. \quad (4.31)$$

To gauge fix the dilatations, one impose the *D-gauge*:

$$-i(X^I \bar{F}_I - F_I \bar{X}^I) = 1 . \quad (4.32)$$

To fix the chiral  $U(1)$  symmetry, one fixes the *A-gauge*:

$$X^0 = \bar{X}^0 . \quad (4.33)$$

Due to these constraints on the scalars, the Poincaré supergravity has only  $N_V$  independent scalars.

In order to fix the  $S$ -supersymmetry, one imposes another gauge called the *S-gauge*. This constraint can be solved by eliminating one of the vector multiplet fermions. This gauge also breaks  $Q$ -supersymmetry, but a combination of the  $S$  and  $Q$  supersymmetries is preserved and corresponds to the physical supertransformations in the Poincaré theory.

Finally, to fix the local  $SU(2)$  symmetry, one imposes the *V-gauge*:

$$\Phi_\alpha^i = \delta_\alpha^i \quad (4.34)$$

At each step in the gauge fixing process, one has to be careful to respect the previous gauge choices, and this leads to compensating field dependent transformations in the rules for the various remaining transformations. This is one of the reasons the final theory is more complicated. Finally, one has to solve algebraic equations to get rid of the auxiliary fields  $D$  and  $\chi$ . At the end of this procedure, one gets the  $\mathcal{N} = 2$  Poincaré supergravity with a bosonic Lagrangian:

$$\begin{aligned} 8\pi e^{-1} \mathcal{L} = & (-i(X^I \bar{F}_I - F_I \bar{X}^I)) \cdot (-\frac{1}{2} R) \\ & + [i \nabla_\mu F_I \nabla^\mu \bar{X}^I + \frac{1}{4} i F_{IJ} (F_{ab}^{-I} - \frac{1}{4} \bar{X}^I T_{ab}^{ij} \varepsilon_{ij}) (F^{-abJ} - \frac{1}{4} \bar{X}^J T_{ab}^{ij} \varepsilon_{ij}) \\ & - \frac{1}{8} i F_I (F_{ab}^{+I} - \frac{1}{4} X^I T_{abij} \varepsilon^{ij}) T_{ab}^{ij} \varepsilon_{ij} - \frac{1}{8} i F_{IJ} Y_{ij}^I Y^{Jij} - \frac{i}{32} F (T_{abij} \varepsilon^{ij})^2 \\ & + \frac{1}{2} i F_{\hat{A}} \hat{C} - \frac{1}{8} i F_{\hat{A}\hat{A}} (\varepsilon^{ik} \varepsilon^{jl} \hat{B}_{ij} \hat{B}_{kl} - 2 \hat{F}_{ab}^- \hat{F}_{ab}^-) + \frac{1}{2} i \hat{F}^{-ab} F_{\hat{A}I} (F_{ab}^{-I} - \frac{1}{4} \bar{X}^I T_{ab}^{ij} \varepsilon_{ij}) \\ & - \frac{1}{4} i \hat{B}_{ij} F_{\hat{A}I} Y^{Iij} + \text{h.c.}] \\ & - i(X^I \bar{F}_I - F_I \bar{X}^I) \cdot (\nabla^a V_a - \frac{1}{2} V^a V_a - \frac{1}{4} |M_{ij}|^2 + D^a \Phi_\alpha^i D_a \Phi_\alpha^i) . \end{aligned} \quad (4.35)$$

Note that both the covariant derivatives defined above are used in this expression, they are related by

$$D^a V_a = \nabla^a V_a - 2f_a^a + \text{fermionic terms} . \quad (4.36)$$

### 4.3 Localization

We now turn to the evaluation of the supersymmetric black hole functional integral defined in §3.2 using the localization techniques discussed in §4.1. We use the formalism of §4.2 so that the supercharge used for localization is realized off-shell.

The on-shell equations of motion that follow from the above Lagrangian (4.35) admit a half-BPS black hole solution [91, 92, 93, 90]. The near horizon geometry is an  $AdS_2 \times S^2$  which admits eight conformal supersymmetries<sup>8</sup>. The values of other fields are determined by the attractor mechanism [94, 95, 96] in terms of the charges consistent with the isometries. The near-horizon  $AdS_2 \times S^2$  geometry with the attractor values of the other fields can also directly be derived from the BPS equations [97].

We first review this on-shell solution in §4.3.1. We then proceed to find the localizing instanton solution in §4.3.2 and evaluate the renormalized action for this solution in §4.3.3. We will sometimes refer to the localizing solution as the off-shell solution since for this solution the scalar fields are excited away from the attractor values inside the  $AdS_2$ . In §4.3.4 we put together these ingredients to reduce the functional integral of  $\widehat{W}(q, p)$  to an ordinary integral on the localizing submanifold.

#### 4.3.1 On-shell attractor geometry

Symmetries of  $AdS_2 \times S^2$  imply that various field in the near horizon region take the form

$$\begin{aligned} ds^2 &= v \left[ -(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} \right] + v \left[ d\psi^2 + \sin^2(\psi)d\phi^2 \right] , \\ F_{rt}^I &= e_*^I, \quad F_{\psi\phi}^I = p^I \sin \psi, \quad X^I = X_*^I, \quad T_{rt}^- = v w , \\ D - \frac{1}{3}R &= 0, \quad M_{ij} = 0, \quad \Phi_i^\alpha = \delta_i^\alpha, \quad Y_{ij}^I = 0 . \end{aligned} \tag{4.37}$$

The values of the constants  $(e_*^I, X_*^I, v_*)$  that appear in this solution are determined in terms of the charges  $(q_I, p^I)$  by the attractor equations which follow from the BPS conditions [91], or, equivalently using the entropy function formalism [89]:

$$v = \frac{16}{\overline{w}w}, \quad \hat{A} = -4\omega^2 , \tag{4.38}$$

$$e_*^I - ip^I - \frac{1}{2}\overline{X}_*^I v w = 0 , \tag{4.39}$$

$$4i(\overline{w}^{-1}\overline{F}_I - w^{-1}F_I) = q_I . \tag{4.40}$$

Taking the real and imaginary parts of (4.39) and substituting (4.38) gives

$$4(\overline{w}^{-1}\overline{X}_*^I + w^{-1}X_*^I) = e_*^I , \tag{4.41}$$

$$4i(\overline{w}^{-1}\overline{X}_*^I - w^{-1}X_*^I) = p^I , \tag{4.42}$$

---

<sup>8</sup>As mentioned above, these conformal supersymmetries are not the conformal supersymmetries of the four-dimensional theory discussed in the last section, the latter are gauge symmetries in that formalism.

where  $F_I$  should be thought of as functions of  $X_*^I$ . This geometry preserves eight superconformal supersymmetries as reviewed §A which extends the symmetries to the supergroup  $SU(1, 1|2) \otimes SU(2)'$  discussed in §4.1.2. The field  $w$  can be fixed by a gauge choice. In the rest of the paper, we choose a gauge in which  $w = \overline{w} = 4$  using the local scaling symmetry of the Lagrangian and the  $U(1)$  invariance. In this gauge, the radius  $v$  of both  $AdS_2$  and  $S^2$  equals one, this simplifies the discussion of Killing spinors<sup>9</sup>.

### 4.3.2 Localizing action and the localizing instantons

In order to use the technique of localization for our system, we need to pick a subalgebra of the full supersymmetry algebra discussed in §4.1.2, whose bosonic generator is compact. We shall choose the subalgebra generated by the action of the supercharge

$$Q_1 = G_+^{++} + G_-^{--} , \quad (4.43)$$

which generates the compact  $U(1)$  action:

$$Q_1^2 = 4(L - J) . \quad (4.44)$$

The explicit form of the Killing spinors can be found in §A. The above choice of the supercharge corresponds to choosing the supersymmetry parameter  $\zeta_1$  defined in (A.18). In this section, we use the notation  $Q \equiv Q_1$ ,  $\zeta \equiv \zeta_1$ .

The localizing Lagrangian is then defined by

$$\mathcal{L}^Q := QV \quad \text{with} \quad V := (Q\Psi, \Psi) , \quad (4.45)$$

where  $\Psi$  refers to all fermions in the theory. The localizing action is then defined by

$$S^Q = \int d^4x \sqrt{g} \mathcal{L}^Q . \quad (4.46)$$

The localization equations that follow from this action are

$$Q\Psi = 0 . \quad (4.47)$$

These are the equations that we would like to solve.

We assume that the supergroup isometries of the near horizon geometry are not broken further by the Weyl multiplet fields. By construction, as long as these symmetries are maintained, the fermions of the Weyl multiplet do not transform under the action of  $Q$  (A.1) –(A.3) in the  $AdS_2$  attractor background. One can check that the fermions of the chiral multiplet and the non-linear multiplet also do not transform in this background. This prompts us to look for solutions where one still has the  $AdS_2$  attractor geometry, but the scalars of the vector

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<sup>9</sup>This is different from the gauge used in the previous section and also from the gauge  $\omega = 8$  which is commonly used. These gauge choices do not affect considerations in this paper, but a better understanding of different gauge choices can be useful to simplify the analysis. We plan to return to this issue in future.

multiplets can move away from their attractor values<sup>10</sup>. As we will see there do exist nontrivial solutions where the vector multiplet fields get excited maintaining the symmetries of the attractor geometry.

The action of  $Q$  on the fermionic field of the vector multiplet takes the form (A.19)

$$Q \Omega_+^{Ii} = \frac{1}{2}(F_{\mu\nu}^{I-} - \frac{1}{4}\overline{X}^I T_{\mu\nu}^-) \gamma^\mu \gamma^\nu \zeta_+^i + 2i\partial X^I \zeta_-^i + Y_j^{Ii} \zeta_+^j, \quad (4.48)$$

$$Q \Omega_-^{Ii} = \frac{1}{2}(F_{\mu\nu}^{I+} - \frac{1}{4}X^I T_{\mu\nu}^+) \gamma^\mu \gamma^\nu \zeta_-^i + 2i\partial \overline{X}^I \zeta_+^i + Y_j^{Ii} \zeta_-^j. \quad (4.49)$$

Let us recall the attractor equations for the constant values of the various fields in terms of the electric gauge field strengths  $e^I$  and the magnetic charges  $p^I$ :

$$e_*^I - ip^I - 2\overline{X}_*^I = 0, \quad e_*^I + ip^I - 2X_*^I = 0 \quad Y_{ij*}^I = 0. \quad (4.50)$$

We are interested in the off-shell solutions in which the vector multiplet scalars  $X^I$  move away from their attractor values  $X_*^I$ . We therefore parametrize the off-shell  $X^I$  fields as

$$X^I := X_*^I + \Sigma^I, \quad \overline{X}^I := \overline{X}_*^I + \overline{\Sigma}^I, \quad (4.51)$$

so that  $\Sigma^I$  and  $\overline{\Sigma}^I$  are values the scalar fields away from the attractor values. We further write

$$\Sigma^I = H^I + iJ^I, \quad \overline{\Sigma}^I = H^I - iJ^I. \quad (4.52)$$

Note that  $Y_{ij}^I = \epsilon_{ik}\epsilon_{jl}Y^{Ikl}$  are triplets under the  $SU(2)$  rotation. It will turn out that for the BPS equations that we solve, they all have to be aligned along the same direction in the  $SU(2)$  space. Hence we parametrize them as

$$Y_1^{I1} = -Y_2^{I2} = K^I; \quad Y_2^{I1} = Y_1^{I2} = 0, \quad (4.53)$$

where we have defined  $Y_j^i = \varepsilon_{jk}Y^{ik}$ . Similarly we parametrize the gauge fields away from the attractor values as

$$F_{\mu\nu}^I = F_{\mu\nu*}^I + f_{\mu\nu}^I. \quad (4.54)$$

With this parametrization, we can add the two equations (4.48) and perform a Euclidean continuation to obtain

$$Q \Omega^{Ii} = \frac{1}{2}f_{ab}^I \gamma^a \gamma^b \zeta^i + 2i\partial H^I \zeta^i + 2\partial J^I \gamma_5 \zeta^i - 2iH^I \gamma^0 \gamma^1 \zeta^i - 2J^I \gamma^2 \gamma^3 \zeta^i + Y_j^{Ii} \zeta^j. \quad (4.55)$$

for the Dirac spinors  $\Omega^{Ii} = \Omega_+^{Ii} + \Omega_-^{Ii}$ . Note that the  $a, b$  are tangent space indices and all gamma matrices  $\gamma^a$  above are constant matrices of Euclidean  $\mathbb{R}^4$ .

The inner product for spinors  $\chi_1$  and  $\chi_2$  in Euclidean space is simply

$$(\chi_1, \chi_2) = \chi_1^\dagger \chi_2. \quad (4.56)$$

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<sup>10</sup>Solutions more general than our simplifying ansatz are in principle possible where the Weyl multiplet fields also vary inside the  $AdS_2$ .



With this inner product, the localization Lagrangian (4.45) restricted to only the vector multiplet fermions is given by

$$\mathcal{L}^Q = QV := Q(Q\Psi, \Psi) \quad (4.57)$$

with  $V$  chosen as in (4.45) with  $\Psi$  denoting the vector multiplet fermions. Note that  $V$  is H-invariant because  $\zeta$  is independent of the combination  $\theta - \phi$  and  $H$  is the vector field that generates translations along  $\theta - \phi$ . The bosonic part of this Lagrangian is

$$\mathcal{L}_{\text{bos}}^Q \equiv QV|_{\text{bosonic}} = \sum_{I=0}^{n_V} (Q\Omega^I, Q\Omega^I) . \quad (4.58)$$

With our choice of the inner product (4.56) this Lagrangian is manifestly positive definite.

The choice of  $Q$  is determined by the choice of the Killing spinor  $\zeta$ . Substituting the explicit form of the Killing spinor  $\zeta$  and the gamma matrices defined in §A, the bosonic Lagrangian  $\mathcal{L}_{\text{bos}}^Q$  as a function of the fields  $H, J, K, f$  can be evaluated after somewhat tedious algebra. We find that  $\frac{1}{2}\mathcal{L}_{\text{bos}}^Q$  equals

$$\begin{aligned} & \cosh(\eta)[K - 2\text{sech}(\eta)H]^2 \\ & + 4\cosh(\eta)[H_1 + H\tanh(\eta)]^2 + 4\cosh(\eta)[H_0^2 + H_2^2 + H_3^2] \\ & + 2A\left[f_{01}^- - J - \frac{1}{A}(\sin(\psi)J_3 - \sinh(\eta)J_1)\right]^2 + 2B\left[f_{01}^+ + J - \frac{1}{B}(\sin(\psi)J_3 + \sinh(\eta)J_1)\right]^2 \\ & + 2A\left[f_{03}^- + \frac{1}{A}(\sin(\psi)J_1 + \sinh(\eta)J_3)\right]^2 + 2B\left[f_{03}^+ + \frac{1}{B}(\sin(\psi)J_1 - \sinh(\eta)J_3)\right]^2 \\ & + 2A\left[f_{02}^- + \frac{1}{A}(\sin(\psi)J_0 + \sinh(\eta)J_2)\right]^2 + 2B\left[f_{02}^+ - \frac{1}{B}(\sin(\psi)J_0 + \sinh(\eta)J_2)\right]^2 \\ & + \frac{4\cosh(\eta)}{AB}[\sinh(\eta)J_0 - \sin(\psi)J_2]^2 + \frac{4\cosh(\eta)\sinh^2(\eta)}{AB}[J_1^2 + J_3^2], \end{aligned} \quad (4.59)$$

where

$$H_a^I := e_a^\mu \partial_\mu H^I, \quad J_a^I := e_a^\mu \partial_\mu J^I, \quad (4.60)$$

and

$$A := \cosh(\eta) + \cos(\psi), \quad B := \cosh(\eta) - \cos(\psi). \quad (4.61)$$

It is understood that in (4.59) all squares are summed over the index  $I$ . Recall that  $a = 0, 1, 2, 3$  correspond to the directions along the coordinates  $\theta, \eta, \phi, \psi$  respectively used for example in (4.8). Since  $A$  and  $B$  are positive,  $\mathcal{L}_{\text{bos}}^Q$  is a sum of positive squares.

The minimization equations now follow by setting each of the squares in (4.59) to zero. This leads to simple first order differential equations for various fields which have to be solved with boundary conditions consistent with the definition of the original functional integral on Euclidean  $AdS_2$  space. Equations (4.51), (4.53), (4.54) imply that fields  $\Sigma^I$  and  $K^I$  and  $f^I$  must vanish at the boundary.

It is easy to see that with these boundary conditions,  $J^I$  and  $f_{ab}^I$  must both vanish throughout space. Setting the first line in (4.59) to zero implies

$$K^I = \frac{2H^I}{\cosh(\eta)} \quad (4.62)$$

Setting the second line in (4.59) to zero leads to differential equations that can be easily solved to obtain

$$H^I = \frac{C^I}{\cosh(\eta)} . \quad (4.63)$$

We have thus succeeded in finding a family of exact solutions to the localization equations which respect the classical boundary conditions on  $AdS_2$  and are smooth everywhere in the interior. In terms of the original variables defined in (4.51), we have

$$X^I = X_*^I + \frac{C^I}{\cosh(\eta)} , \quad \bar{X}^I = \bar{X}_*^I + \frac{C^I}{\cosh(\eta)} \quad (4.64)$$

$$Y_1^{I1} = -Y_2^{I2} = \frac{2C^I}{\cosh(\eta)^2} . \quad (4.65)$$

Since the scalar fields are now excited away from their attractor values, they are no longer at the minimum of the classical entropy function. Even though scalar fields ‘climb up’ the potential away from the minimum of the entropy function the solution remains Q-supersymmetric (in the Euclidean theory) because an auxiliary field gets excited appropriately to satisfy the Killing spinor equations.

It is worth pointing out that the solutions (4.63) and (4.62) look much simpler if we use the conformal transformation (4.13) in §4.1.2 to map  $AdS_2 \times S^2$  to  $S^4$ . Since the scalar fields  $X$  and the auxiliary fields  $Y$  have Weyl weight 1 and 2 respectively, and since the conformal factor is  $\cosh(\eta)$ , the fields  $\Sigma$  and  $Y$  are simply constant on  $S^4$ . This is very similar to the localizing solution found by Pestun [11] in a very different context of computing the expectation value of Wilson line in super Yang-Mills theory on  $S^4$ . Of course, under this conformal transformation the attractor values also will transform and since they are constant on  $AdS_2 \times S^2$ , they will no longer be constant on  $S^4$ . It is therefore more natural to work in the  $AdS_2 \times S^2$  frame. In any case, for computing the quantum entropy, the  $AdS_2$  boundary conditions play an important role as we will see in the next subsection. As pointed out in [12], in this frame our computation has close formal similarity with the gauge theory computation of ‘t Hooft-Wilson line in the formulation of [98, 99] which could be useful in the computation of one-loop determinants and the instanton contributions. Note that we are using localization techniques to evaluate a bulk functional integral of supergravity whereas in [11, 98, 99] it was used to evaluate a functional integral in the boundary gauge theory.

### 4.3.3 Renormalized action for the localizing instantons

To obtain the exact macroscopic quantum partition function we would like to evaluate the renormalized action restricted to the submanifold  $\mathcal{M}_Q$  in field space of localizing instantons.

We will find that even though both the original action and the solution are rather complicated, the renormalized action is a remarkably simple function of the collective coordinates  $\{C^I\}$  determined entirely by the prepotential. Recall that the renormalized action defined in the last section takes the form

$$\mathcal{S}_{\text{ren}} := \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{bdry}} + i \frac{q_I}{2} \int_0^{2\pi} A_\theta^I d\theta . \quad (4.66)$$

The charges used here are related to the ones used in (3.13) by  $q_I = -2q_i$  to be consistent with the normalization of gauge fields used in the literature, for example, in the reviews [68, 87].

We proceed to evaluate the bulk action given as a four dimensional integral of the the supergravity Lagrangian (4.35) over  $AdS_2 \times S^2$ . We note first that since various auxiliary fields vanish for the off-shell solution, the Lagrangian (4.35) simplifies to (recall  $\hat{A} = (T_{ab}^{ij} \varepsilon_{ij})^2$ ):

$$\begin{aligned} 8\pi \mathcal{L} = & -\frac{i}{2} (X^I \bar{F}_I - \bar{X}^I F_I) R \\ & + \left[ i \partial_\mu F_I \partial^\mu \bar{X}^I + \frac{i}{4} F_{IJ} (F_{\mu\nu}^{-I} - \frac{1}{4} \bar{X}^I T_{\mu\nu}^{ij} \varepsilon_{ij}) (F^{-J\mu\nu} - \frac{1}{4} \bar{X}^J T^{\mu\nu ij} \varepsilon_{ij}) \right. \\ & \left. + \frac{i}{8} \bar{F}_I (F_{\mu\nu}^{-I} - \frac{1}{4} \bar{X}^I T_{\mu\nu ij} \varepsilon^{ij}) T_{\mu\nu}^{ij} \varepsilon_{ij} - \frac{i}{8} F_{IJ} Y_{ij}^I Y^{Jij} + \frac{i}{32} \bar{F} \hat{A} + \frac{i}{2} F_{\hat{A}} \hat{C} + \text{h.c.} \right] \end{aligned} \quad (4.67)$$

Moreover, for  $AdS_2 \times S^2$  both the Ricci scalar  $R$  and the Weyl tensor  $C$  are zero. Substituting  $X^I = X_*^I + \Sigma^I$  and  $\bar{X}^I = \bar{X}_*^I + \bar{\Sigma}^I$  from (4.51) and using the attractor equation (4.50) in the form

$$F_{\mu\nu}^{-I} - \frac{1}{4} \bar{X}_*^I T_{\mu\nu}^{ij} \varepsilon_{ij} = 0 , \quad (4.68)$$

we get

$$8\pi \mathcal{L} = i F_{IJ} (\partial_\eta \bar{\Sigma}^I) (\partial_\eta \Sigma^J) - i F_{IJ} \bar{\Sigma}^I \Sigma^J + \frac{i}{4} F_{IJ} K^I K^J + 2i \bar{F}_I \bar{\Sigma}^I - 2i \bar{F} + \text{h.c.} . \quad (4.69)$$

Substituting the solution (4.64) into the above equation, we find that the first three terms add up to zero. We are thus left with

$$8\pi \mathcal{L} = 2i \bar{F}_I \bar{\Sigma}^I - 2i \bar{F} + \text{h.c.} . \quad (4.70)$$

Since we keep the classical values  $X_*^I, \bar{X}_*^I$  fixed in this problem, differentiating with respect to  $X^I$  is the same as differentiating with respect to  $\Sigma^I$ . This can be explicitly evaluated to find

$$8\pi \mathcal{L} = 2i \partial_r (r(F - \bar{F})) , \quad \text{with} \quad \Sigma^I = \frac{C^I}{r} . \quad (4.71)$$

The  $\mathcal{N} = 2$  supergravity Euclidean action is

$$\mathcal{S}_{\text{bulk}} = \int d^4 x \sqrt{g} \mathcal{L} . \quad (4.72)$$

The off-shell fields do not depend on the coordinates of the  $S^2$  and the angular variable  $\theta$  of the  $AdS_2$ . These integrals can be done trivially and give an overall factor of  $8\pi^2$ , so that

$$\begin{aligned}
\mathcal{S}_{\text{bulk}} &= 8\pi^2 \int_0^{\eta_0} \mathcal{L} \sinh(\eta) d\eta = 8\pi^2 \int_1^{r_0} \mathcal{L} dr , \\
&= 2\pi i \int_1^{r_0} dr \partial_r (r(F - \bar{F})) , \\
&= 2\pi i r_0 \left[ F(X_*^I + \frac{C^I}{r_0}) - \bar{F}(X_*^I + \frac{C^I}{r_0}) \right] - 2\pi i \left[ F(X_*^I + C^I) - \bar{F}(X_*^I + C^I) \right].
\end{aligned} \tag{4.73}$$

The first piece in (4.73) which is linear in  $r_0$  can be rewritten as:

$$\begin{aligned}
2\pi i r_0 \left( F(X_*^I + \frac{C^I}{r_0}) - \bar{F}(X_*^I + \frac{C^I}{r_0}) \right) &= \\
&= 2\pi i r_0 (F(X_*^I) - \bar{F}(X_*^I)) + 2\pi i (F_I(X_*^I) - \bar{F}_I(X_*^I)) C^I + \mathcal{O}(1/r_0) \\
&= 2\pi i r_0 (F(X_*^I) - \bar{F}(X_*^I)) - 2\pi q_I C^I + \mathcal{O}(1/r_0)
\end{aligned} \tag{4.74}$$

where we have used a Taylor expansion in the first line and the attractor equation

$$F_I(X_*^I) - \bar{F}_I(X_*^I) = i q_I \tag{4.75}$$

in the second.

The Wilson line evaluates to

$$i \frac{q_I}{2} \int_0^{2\pi} A_\theta^I d\theta = \pi q_I e_*^I (r_0 - 1) . \tag{4.76}$$

Hence we choose

$$\mathcal{S}_{\text{bdry}} = -2\pi r_0 \left( \frac{q_I e_*^I}{2} + i (F(X_*^I) - \bar{F}(X_*^I)) \right) . \tag{4.77}$$

so that  $\mathcal{S}_{\text{ren}} = \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{bdry}} + i \frac{q}{2} \oint A$  is finite.

As explained in section §3, the main purpose of the boundary action is to cancel the divergence in the bare bulk action plus Wilson line which grows linearly with the length of the boundary. In order to cancel this divergence, we use a boundary cosmological constant which must be specified along with the other boundary data. Indeed we have found that  $\mathcal{S}_{\text{bdry}}$  which is a constant that grows linearly with the length of the boundary indeed only depends on the fixed charges and not on the fluctuating fields.

In general, however, there could be a finite part of the boundary action which does depend on the fields that are integrated over. The full boundary action should be constrained by supersymmetry. We shall discuss the supersymmetry of the functional integral in appendix §C. The conclusion of the analysis in appendix §C is quite simple – the finite part of the boundary action in our problem actually vanishes due to supersymmetry, and therefore the

above prescription for  $S_{\text{ren}}$  as a sum of terms (4.73), (4.77) and (4.76) is already supersymmetric. In appendix §C, we shall rewrite the above in a manner that is manifestly supersymmetric. This rewriting takes the form of a functional integral with a supersymmetric Wilson line [100, 101] with the bulk action as above (4.73), and a boundary action which exactly cancels the boundary piece in (4.73).

We thus obtain the following expression for the renormalized action:

$$\mathcal{S}_{\text{ren}} = -\pi q_I e_*^I - 2\pi q_I C^I - 2\pi i (F(X_*^I + C^I) - \overline{F}(X_*^I + C^I)) , \quad (4.78)$$

The notation  $e_*^I$  refers to the classical values of the electric field strengths as a function of the charges  $(q_I, p^I)$ . Using the scalar attractor values (4.41), and the new variable

$$\phi^I \equiv e_*^I + 2C^I , \quad (4.79)$$

we can express the renormalized action in a remarkably simple form:

$$\mathcal{S}_{\text{ren}}(\phi, q, p) = -\pi q_I \phi^I + \mathcal{F}(\phi, p) . \quad (4.80)$$

with

$$\mathcal{F}(\phi, p) = -2\pi i \left[ F\left(\frac{\phi^I + ip^I}{2}\right) - \overline{F}\left(\frac{\phi^I - ip^I}{2}\right) \right] . \quad (4.81)$$

Note that the electric field remains fixed at the attractor value but  $\phi^I$  can still fluctuate with  $C^I$  taking values over the real line. We will discuss the significance of this fact in §4.4. Note also that the prepotential is evaluated at precisely for values of the scalar fields at the origin of  $AdS_2$  and not at the boundary of  $AdS_2$ . Thus the classical contribution to the localization integrand will be of the form

$$e^{\mathcal{S}_{\text{ren}}} = e^{-\pi \phi^I q_I + \mathcal{F}(\phi, p)} \quad (4.82)$$

There will be additional contributions to the integral which we discuss next.

#### 4.3.4 Evaluation of $\widehat{W}(q, p)$

We have thus determined which field configuration to integrate over and the classical action for these configuration. The full functional integral will require three additional ingredients.

- The integration measure over the  $\{C^I\}$  fields over the submanifold  $\mathcal{M}_Q$  of critical points of  $Q$  simply descends from the measure  $\mu$  of supergravity over the field space  $\mathcal{M}$ . We denote this measure by  $[dC]_\mu$  which can be computed using standard methods of collective coordinate quantization.
- There will be one-loop determinants of fluctuations around the localizing manifold which can be evaluated from the quadratic piece of the localizing action  $S^Q$ . We denote this determinant contribution by  $Z_{\text{det}}$ . It is in principle a straightforward but technically involved computation. Very similar determinants have been analyzed in detail for gauge theory [11]. In string theory, the one-loop determinants and the duality invariance measure around the on-shell solution have been analyzed in [75] and in [102, 103] respectively.

Some aspects of these computations both from gauge theory and from around the on-shell saddle point could be adapted to study the measure and determinants around our off-shell instantons solutions [19].

- In addition, there will be a contribution from point instantons and anti-instantons viewed as singular configurations that couple to the vector multiplet fields as long as they preserve the same supersymmetry. In gauge theory computations [11], the instantons will be localized at the center of  $AdS_2$  and at the north pole of the  $S^2$  whereas the anti-instantons will be localized at the center of  $AdS_2$  and at the south pole of the  $S^2$ . Since string theory contains gauge theory at low energies we expect a similar structure also in string theory. We denote this generating function for the instantons by  $Z_{inst}$ . The generating function for anti-instantons will be the complex conjugate of the generating function for instantons. We will thus get a factor of  $|Z_{inst}|^2$  which will depend on the details of the string compactification, the spectrum of wrapped brane-instantons, and the duality frame under consideration. In gauge theory this generating function is the equivariant instanton partition function computed by Nekrasov [104]. Since the low energy limit of string theory will reduce to gauge theory on  $AdS_2 \times S^2$ , it would be interesting to explore if there are generalization of the gauge theory results to string theory.

Putting these ingredients together we can conclude that the functional integral will have the form

$$\widehat{W}(q, p) = \int_{\mathcal{M}_Q} e^{-\pi \phi^I q_I} e^{\mathcal{F}(\phi, p)} |Z_{inst}|^2 Z_{det} [dC]_\mu. \quad (4.83)$$

Note that the Wilson line  $\widehat{W}$  corresponds to localization on  $AdS_2 \times S^2$  while  $W(q, p)$  can contain additional contributions coming from orbifolds. We have thus successfully reduced the functional integral to ordinary integrals. The dominant piece of the answer given by  $e^{-S_{ren}}$  we have already evaluated explicitly.

In specific string compactifications the undetermined factors  $Z_{det}$  and  $|Z_{inst}|^2$  can simplify. For example, with  $\mathcal{N} = 4$  supersymmetry, in gauge theory both  $|Z_{instanton}|^2$  and  $Z_{det}$  equal unity. Similarly, it was found in [75] that very similar determinant factors for vector multiplets equal unity  $\mathcal{N} = 4$  theories. One expects that this simplification will extend to the factors appearing in (4.83) around the localizing solution in  $\mathcal{N} = 4$  theories.

#### 4.4 Quantum entropy and the topological string

We now turn to the original problem of evaluating of  $W(q, p)$ . There are several issues that have to be addressed to extend the supergravity computation to a full string computation.

- First, the full action of string theory of course contains more fields in addition to vector multiplets, in particular the hyper multiplets.
- Second, even if we restrict our attention to vector multiplets, the action will in general contain not just the F-terms which are chiral superspace integrals but also the D-terms which are nonchiral superspace integrals.

- Third, there can be additional contributions from functional integral over orbifolds of  $AdS_2$  that are allowed in the full string theory but not visible in supergravity.

We discuss these questions below.

#### 4.4.1 D-terms, hyper-multiplets, and evaluation of $W(q, p)$

We have thus far considered only F-type terms for the action of the vector multiplets which are chiral integrals over  $\mathcal{N} = 2$  superspace of the form  $\int d^4\theta$ . The effective action of string theory will contain in general D-type terms which are nonchiral integrals over  $\mathcal{N} = 2$  superspace of the form  $\int d^4\theta d^4\bar{\theta}$ . It is not *a priori* clear that these terms will not contribute to the functional integral. We would like to make the following two observations in this connection.

- Since our localizing action  $S^Q$  follows from off-shell supersymmetry transformations, it does not depend on what terms are present in the physical action  $S$ . Hence our localizing instanton solutions are universal and they will continue to exist even with the addition of the D-terms. The question then reduces to evaluating the D-terms on these solutions to obtain their contribution to the renormalized action.
- It has recently been shown [105] that a large class of D-type terms do not contribute to the Wald entropy. This class of terms are constructed using the ‘kinetic multiplet’  $T$  obtained from a chiral multiplet  $\Phi$  of Weyl weight 0 by  $T = \overline{D}^4\Phi$  which transforms like a chiral multiplet of Weyl weight 2. One can construct now supersymmetry invariant terms in the action as chiral integrals  $\int d^4\theta$  with arbitrary polynomials involving the kinetic multiplet and other chiral multiplets. Since four antichiral derivatives have the same effect as the four antichiral integrals, these terms correspond to D-terms with non chiral integrals  $\int d^4\theta d^4\bar{\theta}$  of terms involving the original field  $\Phi$ . The nonrenormalization theorem of [105] shows that D-terms of this type do not contribute to the Wald entropy. Since the renormalized action of the localizing instantons follows from the bulk action and has the same form as the entropy function, it should be possible to extend this nonrenormalization theorem to the renormalized action discussed in this paper.

These two points indicate that the D-terms, or at least a large subclass of them, may in fact not contribute to the renormalized action.

Adding hyper multiplets does not change the transformation rules of the vector multiplets. We therefore expect that the localizing instantons that we have found here will continue to exist. There could be in principle additional localizing solutions where hyper multiplet fields are excited but this may not necessarily happen. It then only remains to check that the coupling of hyper multiplets and vector multiplets at high order cannot contribute to the renormalized action. Lacking an offshell formulation of couplings between hypers and vectors, we cannot at present address this question but perhaps something analogous to the nonrenormalization theorem discussed above can be extended to these terms as well.

In any case, these questions can be systematically investigated in the context of our off-shell localizing instantons. If some of the D-terms do happen to contribute to the renormalized

action, their contribution can be taken into account by evaluating them on the off-shell solutions. Similarly if there are new localizing instantons upon the inclusion of hypers, those too can be added as separate contribution to the final answer for the functional integral.

If the hyper multiplets and D-terms can be ignored for reasons outlined above, one can conclude that  $W_0(q, p)$  has the same form as  $\widehat{W}(q, p)$  evaluated in §4.3

$$W_0(q, p) = \int_{\mathcal{M}_Q} e^{-\pi\phi^I q_I} |Z_{top}(\phi, p)|^2 Z_{det}[dC]_\mu \quad (4.84)$$

The contribution from the orbifolds of  $AdS_2$  also has a very similar structure since the localizing instanton solution is still valid.

## 5. Quantum entropy of large black holes in IIB on $T^6$

The  $AdS_2/CFT_1$  correspondence thus provides a simple and yet nontrivial example of holography. Note that  $d(q, p)$  is the statistical degeneracy of the ensemble of quantum microstates that correspond to the black hole which in general is a highly nontrivial function of the integer charges. If the black hole preserves at least four supersymmetries this degeneracy equals an index and hence can be computed reliably in several examples. On the other hand,  $W(q, p)$  is the generalization of the exponential of the Wald entropy of the black hole. The equality of  $d(q, p)$  and  $W(q, p)$  for arbitrary finite values of the charges<sup>11</sup> can thus be viewed as a statistical interpretation of the *exact* quantum entropy of the black hole for finite charges, including all corrections—both perturbative as well as nonperturbative in  $1/Q$  where  $Q$  denotes a generic charge. It would be a rather nontrivial check of the nonperturbative structure of string theory if  $W(q, p)$  evaluated from a functional integral of string theory can precisely reproduce the full functional dependence of this integer  $d(q, p)$  on the integral charges  $(q, p)$ .

In this section we apply the previous results in the concrete context of supersymmetric black holes preserving four supersymmetries in  $\mathcal{N} = 8$  supersymmetric compactifications of string theory to four spacetime dimensions. Since the structure of the  $\mathcal{N} = 8$  theory is particularly simple, it enables us to analytically perform the ordinary integrals that remain after localization and evaluate  $W(q, p)$  even after including nonperturbative effects. The resulting  $W(q, p)$  matches in remarkable details with the quantum degeneracies  $d(q, p)$  of these black holes that are known independently [19].

### 5.1 Microscopic Quantum Partition Function

Consider Type-II string compactified on a 6-torus  $T^6$ . The resulting four-dimensional theory has  $\mathcal{N} = 8$  supersymmetry with 28 massless  $U(1)$  gauge fields. A charged state is therefore characterized by 28 electric and 28 magnetic charges which combine into the **56** representation

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<sup>11</sup>Rules of  $AdS_2/CFT_1$  correspondence suggest that the natural ensemble to make this comparison is the microcanonical ensemble fixing all charges rather than chemical potentials [8, 9].



of the U-duality group  $E_{7,7}(\mathbb{Z})$ . Under the  $SO(6,6;\mathbb{Z})$  T-duality group, the 28 gauge fields decompose as

$$\mathbf{28} = \mathbf{12} + \mathbf{16} \quad (5.1)$$

where the fields in the vector representation  $\mathbf{12}$  come from the NS-NS sector, while the fields in the spinor representation  $\mathbf{16}$  come from the R-R sector. We obtain an  $\mathcal{N} = 4$  reduction of this theory by dropping four gravitini multiplets. Since each gravitini multiplet of  $\mathcal{N} = 4$  contains four gauge fields, this amounts to dropping sixteen gauge fields which we take to be the R-R fields in the above decomposition. The U-duality group of the reduced theory is

$$SO(6,6;\mathbb{Z}) \times SL(2,\mathbb{Z}) \quad (5.2)$$

where  $SL(2,\mathbb{Z})$  is the electric-magnetic S-duality group.

### 5.1.1 Charge Configuration

We will be interested in one-eighth BPS dyonic states in this theory which preserve four of the thirty-two supersymmetries. To simplify things, we consider the 6-torus to be the product  $T^4 \times S^1 \times \tilde{S}^1$  of a 4-torus and two circles. Let  $n$  and  $w$  be the momentum and winding along the circle  $S^1$ , and  $K$  and  $W$  be the corresponding Kaluza-Klein monopole and NS5-brane charges. Let  $\tilde{n}, \tilde{w}, \tilde{K}, \tilde{W}$  be the corresponding charges associated with the circle  $\tilde{S}^1$ . A general charge vector with these charges can be written as a doublet of  $SL(2,\mathbb{Z})$

$$\Gamma = \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \tilde{n} & n & \tilde{w} & w \\ \tilde{W} & W & \tilde{K} & K \end{bmatrix}_{B'}, \quad (5.3)$$

where the subscript  $B'$  denotes a particular Type-IIB duality frame. The T-duality invariants for this configuration are [106]

$$Q^2 = 2(nw + \tilde{n}\tilde{w}), \quad P^2 = 2(KW + \tilde{K}\tilde{W}), \quad Q \cdot P = nK + \tilde{n}\tilde{K} + wW + \tilde{w}\tilde{W}, \quad (5.4)$$

and the quartic U-duality invariant can be written as

$$\Delta = Q^2 P^2 - (Q \cdot P)^2. \quad (5.5)$$

For our purposes it will suffice to excite only five charges

$$\Gamma = \begin{bmatrix} 0 & n & 0 & w \\ \tilde{W} & W & \tilde{K} & 0 \end{bmatrix}_{B'} \quad (5.6)$$

so that the T-duality invariants are all nonzero. There are three other duality frames that are of interest.

- Frame  $B$ : In this frame the charge configuration becomes

$$\Gamma = \begin{bmatrix} 0 & n & 0 & \tilde{K} \\ Q_1 & \tilde{n} & Q_5 & 0 \end{bmatrix}_B, \quad (5.7)$$

where  $Q_1$  is the number of D1-branes wrapping  $S^1$  and  $Q_5$  is the number of D5-branes wrapping  $T^4 \times S^1$ . This frame is particularly useful for the microscopic derivation of the degeneracies described in §5.1.2. With  $\tilde{K} = 1$ , the Kaluza-Klein monopole interpolates between  $\mathbb{R}^3 \times \tilde{S}^1$  at asymptotic infinity and  $\mathbb{R}^4$  at the center. The momentum  $\tilde{n}$  at infinity becomes angular momentum at the center. This allows for a 4d-5d lift [46, 26] to relate the degeneracies of the four-dimensional state to those of five-dimensional D1-D5 system carrying momentum  $n$  and angular momentum  $\tilde{n}$ .

- Frame  $A$ : In this frame the charge configuration becomes

$$\Gamma = \begin{bmatrix} 0 & q_0 & 0 & -p^1 \\ p^2 & q_2 & p^3 & 0 \end{bmatrix}_A, \quad (5.8)$$

where  $q_0$  is the number of D0-branes,  $q_2$  is the number of D2-branes wrapping  $S^1 \times \tilde{S}^1$ ,  $p^1$  is a D4-brane wrapping  $T^4$ ,  $p^2$  is a D4-brane wrapping  $\Sigma_{67} \times S^1 \times \tilde{S}^1$  and  $p^3$  is a D4-brane wrapping  $\Sigma_{89} \times S^1 \times \tilde{S}^1$  where  $\Sigma_{ij}$  is a 2-cycle in  $T^4$  along the directions  $ij$ . We will use this frame for localization in §5.2 and §5.4.

- Frame  $B''$ : In this frame the charge configuration becomes

$$\Gamma = \begin{bmatrix} 0 & n & 0 & Q_5 \\ Q_3 & Q_1 & Q_3 & 0 \end{bmatrix}_{B''}, \quad (5.9)$$

where all D-branes wrap the circle  $S^1$  and an appropriate cycle in the  $T^4$ .

We can choose a charge configuration which is even simpler:

$$\Gamma = \begin{bmatrix} 0 & n & 0 & 1 \\ 1 & \nu & 1 & 0 \end{bmatrix} \quad (5.10)$$

where  $n$  is a positive integer and  $\nu$  takes values 0 or 1. The U-duality invariant is

$$\Delta = 4n - \nu^2. \quad (5.11)$$

It is clear that  $\nu = \Delta$  modulo 2, and so these states are completely specified by  $\Delta$ . The states preserve four of the thirty-two supersymmetries. We will henceforth denote the degeneracies of these one-eighth BPS-states with charges (5.10) by  $d(\Delta)$  instead of  $d(q, p)$ .

We should emphasize that a large class of states with the same value of  $\Delta$  can be mapped by U-duality to the state (5.10) considered here but that does not exhaust all states. Note that the invariant  $\Delta$  is the unique quartic invariant of the continuous duality group  $E_{7,7}(\mathbb{R})$  but in general there are additional arithmetic duality invariants of the arithmetic group  $G(\mathbb{Z})$  that

cannot be written as invariants of  $G(\mathbb{R})$ . As a result, not all states with the same value of  $\Delta$  are related by duality. Classification of arithmetic invariants of  $G(\mathbb{Z})$  is a subtle number-theoretic problem. For example, for the  $\mathcal{N} = 4$  compactification where the duality group  $O(22, 6; \mathbb{Z}) \times SL(2, \mathbb{Z})$ , essentially the only relevant arithmetic invariant is given by  $I = \gcd(Q \wedge P)$ ; and the degeneracies are known for all values of  $I$  [49, 50, 29, 30]. To our knowledge a similar complete classification of  $E_{7,7}(\mathbb{Z})$  invariants is not known at present. This would be a problem if one wishes to use canonical or a mixed ensemble. For our purposes, since we will working in the microcanonical ensemble, it will suffice to know the degeneracies for the states in the duality orbit of (5.10).

### 5.1.2 Microscopic Counting

The degeneracies of the 1/8-BPS dyonic states in the type II string theory on a  $T^6$  are given in terms of the Fourier coefficients of the following counting function [27, 107, 42]:

$$F(\tau, z) = \frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)}. \quad (5.12)$$

where  $\vartheta_1$  is the Jacobi theta function and  $\eta$  is the Dedekind function. With  $q := e^{2\pi i\tau}$  and  $y := e^{2\pi iz}$ , they have the product representations

$$\begin{aligned} \vartheta_1(\tau, z) &= q^{\frac{1}{8}}(y^{\frac{1}{2}} - y^{-\frac{1}{2}}) \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^n)(1 - y^{-1}q^n), \\ \eta(\tau) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \end{aligned} \quad (5.13)$$

The derivation of the counting function is simplest in the  $B$  frame (5.7) where we have a D1-D5 system in the background of a single Kaluza-Klein monopole. By the 4d-5d lift, the momentum  $\nu$  can be interpreted as 5d angular momentum. The counting problem essentially reduces to counting bound states in *five* dimensions of a single D1-brane bound to a single D5-brane carrying  $n$  units of momentum and  $\nu$  units of angular momentum. Since the D1-brane can move inside the  $D5$  anywhere on the  $T^4$ , the moduli space of this motion is  $T^4$ . The function  $F$  is the generalized elliptic genus of the corresponding superconformal field theory with target space  $T^4$ . This is evident from the product representation which can be seen as coming from four bosons and four fermions.

Analysis of the Fourier coefficients of  $F$  simplifies enormously by the fact that  $F$  is a *weak Jacobi form*. We recall below a few relevant facts about Jacobi forms [108].

1. *Definition:* A Jacobi form of weight  $k$  and index  $m$  is a holomorphic function  $\varphi(\tau, z)$  from  $\mathbb{H} \times \mathbb{C}$  to  $\mathbb{C}$  which is “modular in  $\tau$  and elliptic in  $z$ ” in the sense that it transforms under the modular group as

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}} \varphi(\tau, z) \quad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{Z}) \quad (5.14)$$

and under the translations of  $z$  by  $\mathbb{Z}\tau + \mathbb{Z}$  as

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \varphi(\tau, z) \quad \forall \quad \lambda, \mu \in \mathbb{Z}, \quad (5.15)$$

where  $k$  is an integer and  $m$  is a positive integer.

2. *Fourier expansion:* Equations (5.14) include the periodicities  $\varphi(\tau + 1, z) = \varphi(\tau, z)$  and  $\varphi(\tau, z + 1) = \varphi(\tau, z)$ , so  $\varphi$  has a Fourier expansion

$$\varphi(\tau, z) = \sum_{n,r} c(n, r) q^n y^r, \quad (q := e^{2\pi i \tau}, y := e^{2\pi i z}). \quad (5.16)$$

Equation (5.15) is then equivalent to the periodicity property

$$c(n, r) = C_r(4nm - r^2), \quad \text{where } C_r(D) \text{ depends only on } r \bmod 2m. \quad (5.17)$$

The function is called a *weak* Jacobi form if it satisfies the condition

$$c(n, r) = 0 \quad \text{unless} \quad n \geq 0. \quad (5.18)$$

3. *Theta expansion:* The transformation property (5.15) implies a Fourier expansion of the form

$$\varphi(\tau, z) = \sum_{\ell \in \mathbb{Z}} q^{\ell^2/4m} h_\ell(\tau) e^{2\pi i \ell z} \quad (5.19)$$

where  $h_\ell(\tau)$  is periodic in  $\ell$  with period  $2m$ . In terms of the coefficients (5.17) we have

$$h_\ell(\tau) = \sum_D C_\ell(D) q^{D/4m} \quad (\ell \in \mathbb{Z}/2m\mathbb{Z}). \quad (5.20)$$

Because of the periodicity property of  $h_\ell$ , equation (5.19) can be rewritten in the form

$$\varphi(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau) \vartheta_{m,\ell}(\tau, z), \quad (5.21)$$

where  $\vartheta_{m,\ell}(\tau, z)$  denotes the standard index  $m$  theta function

$$\vartheta_{m,\ell}(\tau, z) := \sum_{\substack{\lambda \in \mathbb{Z} \\ \lambda \equiv \ell \pmod{2m}}} q^{\lambda^2/4m} y^\lambda = \sum_{n \in \mathbb{Z}} q^{m(n+\ell/2m)^2} y^{\ell+2mn} \quad (5.22)$$

This is the theta expansion of  $\varphi$ . The vector  $h := (h_1, \dots, h_{2m})$  transforms like a modular form of weight  $k - \frac{1}{2}$  under  $SL(2, \mathbb{Z})$ .

With these definitions,  $F(\tau, z)$  is a weak Jacobi form of weight  $-2$  and index  $1$ . The indexed degeneracies for a state carrying  $n$  units of momentum and  $r$  units of angular momentum is then given by  $c(n, r)$  in the Fourier expansion (5.16) of  $F$ . Using (5.17) for  $m = 1$ , we see that  $c(n, r)$  depend only on  $D = 4n - r^2$  and  $r \bmod 2$  which in this case equals  $D \bmod 2$ . Hence,

all information about the Fourier coefficients  $c(n, r)$  of  $F$  is contained in a single function of  $D$  alone which we denote by  $C(D)$ . Our task is thus reduced to determining  $C(D)$  given (5.12).

To read off  $C(D)$  more systematically we use the theta expansion

$$F(\tau, z) = h_0(\tau) \vartheta_{1,0}(\tau, z) + h_1(\tau) \vartheta_{1,1}(\tau, z). \quad (5.23)$$

The functions  $h_\ell(\tau)$  in this case are given explicitly by:

$$h_0(\tau) = -\frac{\vartheta_{1,1}(\tau, 0)}{\eta^6(\tau)} = -2 - 12q - 56q^2 - 208q^3 \dots \quad (5.24)$$

$$h_1(\tau) = \frac{\vartheta_{1,0}(\tau, 0)}{\eta^6(\tau)} = q^{-\frac{1}{4}}(1 + 8q + 39q^2 + \dots) \quad (5.25)$$

For even and odd  $D$ , the coefficients  $C(D)$  can be read off from these expansions of  $h_0$  and  $h_1$  respectively using (5.20).

It is clear that  $D$  can be identified with the duality invariant  $\Delta$  in (5.11). The degeneracies are then given in terms of  $C(D)$  by

$$d(\Delta) = (-1)^{\Delta+1} C(\Delta). \quad (5.26)$$

The factor of  $(-1)^\Delta$  arises because the state in five dimensional spacetime is fermionic for odd  $\Delta$  and contributes to the index with a minus sign. The overall minus sign arises in relating the 4d degeneracies to the 5d degeneracies using the 4d-5d lift [107, 42].

### 5.1.3 Index, Degeneracy, and Fermions

The first few terms in the Fourier expansion of  $F$  are given by

$$F(\tau, z) = \frac{(y-1)^2}{y} - 2 \frac{(y-1)^4}{y^2} q + \frac{(y-1)^4(y^2-8y+1)}{y^3} q^2 + \dots, \quad (5.27)$$

In Table (1) we tabulate the coefficients  $C(\Delta)$  for the first few values of  $\Delta$ .

**Table 1:** Some Fourier coefficients

$\Delta$	-1	0	3	4	7	8	11	12	15
$C(\Delta)$	1	-2	8	-12	39	-56	152	-208	513

It is striking that the sign of  $C(\Delta)$  is alternating. This implies from (5.26) that the degeneracies  $d(\Delta)$  are always positive. This is, in fact, true not only for the first leading coefficients but for all Fourier coefficients, as can be seen from the equations (5.23)–(5.25). Mathematically, the alternating sign of the Fourier coefficients is a somewhat nontrivial property of the specific Jacobi form (5.12) under consideration [109]. Physically, the positivity of  $d(\Delta)$  is even more surprising. After all, these are *indexed* degeneracies corresponding to a spacetime helicity supertrace for a complicated bound states of branes. There is no *a priori* microscopic reason why these should be all positive.

Holography gives a simple physical explanation of the positivity [73, 110]. The near-horizon  $AdS_2$  geometry has an  $SU(1, 1)$  symmetry. If the black hole geometry leaves at least four supersymmetries unbroken, then closure of the supersymmetry algebra requires that the near horizon symmetry must contain the supergroup  $SU(1, 1|2)$ . This implies that such a supersymmetric horizon must have  $SU(2)$  symmetry which can be identified with spatial rotations. If  $J$  is a Cartan generator of this  $SU(2)$ , then for a classical black hole with spherical symmetry, this could mean (depending on the ensemble) that either  $J$  is zero or the chemical potential conjugate to  $J$  is zero. As explained earlier, the  $AdS_2$  path integral naturally fixes the charges and not the chemical potentials and hence  $J = 0$ . Together, this implies

$$\text{Tr}(1) = \text{Tr}(-1)^J, \quad (5.28)$$

that is, index equals degeneracy and must be positive. For a more detailed discussion see [35].

Note the the index equals degeneracy only for the horizon degrees of freedom, but usually one does not compute the index of the horizon degrees of freedom directly. It is easier to compute the index of the asymptotic states as a spacetime helicity supertrace which receives contribution also from the degrees of freedom external to the horizon. It is crucial that the contribution of these external modes is removed from the helicity supertrace before checking the equality (5.28). Typically, modes localized outside the horizon come from fluctuations of supergravity fields and can carry NS-NS charges such as the momentum but not D-brane charges [69, 70]. In a given frame such as the  $A$  frame where all charges come from D-branes, one expects that the Fourier coefficients of  $F(\tau, z)$  will give the degeneracies of only the horizon degrees of freedom.

For the Wilson line expectation value (3.14) the equality (5.28) implies that the functional integral with periodic boundary conditions for the fermions must equal the functional integral with antiperiodic boundary conditions. This is possible for the following reason. All fermionic fields have nonzero  $J$  and couple to the Kaluza-Klein gauge field coming from the dimensional reduction on the  $S^2$ . As discussed above, the microcanonical boundary conditions for the functional integral instructs us to integrate over all the fluctuations of the constant mode. By a change of variables in the functional integral, one can change the origin of the constant mode of the gauge field, and therefore the periodic and antiperiodic boundary conditions for the fermionic fields are equivalent.

### 5.1.4 Rademacher Expansion

One can make very good estimates of Fourier coefficients of a modular form using an expansion due to Hardy and Ramanujan. The leading term of this expansion gives the Cardy formula. A generalization due to Rademacher [111] in fact gives an *exact* expansion for these coefficients in terms of the coefficients of the polar terms *i.e.* terms with  $D < 0$ .

One can apply these methods to the Fourier coefficients of the vector valued modular form  $\{h_l\}$  ( $l = 0, \dots, 2m - 1$ ) of negative weight  $-w$  to obtain [112, 113] a Rademacher expansion for

the coefficients  $C_\ell(D)$  (5.20)

$$C_\ell(D) = (2\pi)^{2-w} \sum_{c=1}^{\infty} c^{w-2} \sum_{\tilde{\ell} \in \mathbb{Z}/2m\mathbb{Z}} \sum_{\tilde{D} < 0} C_{\tilde{\ell}}(\tilde{D}) K(D, \ell, \tilde{D}, \tilde{\ell}; c) \left| \frac{\tilde{D}}{4m} \right|^{1-w} \tilde{I}_{1-w} \left[ \frac{\pi}{c} \sqrt{|\tilde{D}|D} \right],$$

where

$$\tilde{I}_\rho(z) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\sigma}{\sigma^{\rho+1}} \exp\left[\sigma + \frac{z^2}{4\sigma}\right] \quad (5.29)$$

is called the modified Bessel function of index  $\rho$ . This is related to the standard Bessel function of the first kind  $I_\rho(z)$  by

$$\tilde{I}_\rho(z) = \left(\frac{z}{2}\right)^{-\rho} I_\rho(z). \quad (5.30)$$

The sum over  $(\tilde{\ell}, \tilde{D})$  picks up a contribution  $C_{\tilde{\ell}}(\tilde{D})$  from every non-zero term  $q^{\tilde{D}}$  with  $\tilde{D} < 0$  in  $h_{\tilde{\ell}}(\tau)$  (5.20). The coefficients  $K_\ell(D, \ell, \tilde{D}, \tilde{\ell}; c)$  are generalized Kloosterman sums. For  $c > 1$  it is defined as

$$K(D, \ell; \tilde{D}, \tilde{\ell}; c) := e^{-\pi i w/2} \sum_{\substack{-c \leq d < 0 \\ (d, c)=1}} e^{2\pi i \frac{d}{c}(D/4m)} M(\gamma_{c,d})_{\ell\tilde{\ell}}^{-1} e^{2\pi i \frac{a}{c}(\tilde{D}/4m)}, \quad (5.31)$$

where

$$\gamma_{c,d} = \begin{pmatrix} a(ad-1)/c & \\ c & d \end{pmatrix} \quad (5.32)$$

is an element of  $SL(2, \mathbb{Z})$  and  $M(\gamma)$  is the matrix representation of  $\gamma$  on the vector space spanned by the  $\{h_l\}$ . Note that it follows from (5.32) that  $ad = 1 \pmod{c}$ .

The Jacobi form  $F(\tau, z)$  has weight  $-2$  and index  $m = 1$ , so its theta expansion gives a two-component vector  $\{h_0, h_1\}$  of modular forms of weight  $w = -5/2$ . Since there is only a single polar term ( $\tilde{\ell} = 1, \tilde{D} = -1$ ), the Rademacher expansion takes the form:

$$C(D) = 2\pi \left(\frac{\pi}{2}\right)^{7/2} \sum_{c=1}^{\infty} c^{-9/2} K_c(D) \tilde{I}_{7/2}\left(\frac{\pi\sqrt{D}}{c}\right), \quad (5.33)$$

where the Kloosterman sum  $K_c(D)$  is defined by

$$K_c(D) := e^{5\pi i/4} \sum_{\substack{-c \leq d < 0; \\ (d, c)=1}} e^{2\pi i \frac{d}{c}(D/4)} M(\gamma_{c,d})_{\ell 1}^{-1} e^{2\pi i \frac{a}{c}(-1/4)} \quad (c > 1) \quad (5.34)$$

with  $\ell = D \pmod{2}$  and  $ad = 1 \pmod{c}$ .

Under the  $SL(2, \mathbb{Z})$  generators, the modular form  $h_\ell(\tau)$  transform as

$$h_0(\tau + 1) = h_0(\tau), \quad h_0(-1/\tau) = \frac{1+i}{2} \tau^{-5/2} (h_0(\tau) + h_1(\tau)); \quad (5.35)$$

$$h_1(\tau + 1) = -i h_1(\tau), \quad h_1(-1/\tau) = \frac{1+i}{2} \tau^{-5/2} (h_0(\tau) - h_1(\tau)). \quad (5.36)$$

From these transformations, we can read off the matrices  $M(\gamma)$  for the generators  $S$  and  $T$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (5.37)$$

to be

$$M(T) = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \quad M(S) = \frac{e^{\pi i/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (5.38)$$

Using the expression for a general  $SL(2, \mathbb{Z})$  matrix  $\gamma$  in terms of the generators  $S$  and  $T$ , and the representation (5.38), we can obtain the representation  $M(\gamma)$ .

We see from (5.33) that the microscopic degeneracy is an infinite sum of the form

$$d(\Delta) = \sum_{c=1}^{\infty} d_c(\Delta). \quad (5.39)$$

where each term is given by

$$d_c(\Delta) = (-1)^{\Delta+1} 2\pi \left(\frac{\pi}{\Delta}\right)^{7/2} I_{\frac{7}{2}}\left(\frac{\pi\sqrt{\Delta}}{c}\right) \frac{1}{c^{9/2}} K_c(\Delta). \quad (5.40)$$

It is easy to check that

$$K_1 = (-1)^{\Delta+1} \frac{1}{\sqrt{2}}. \quad (5.41)$$

We will see that the Wilson line from the macroscopic side also naturally has the same expansion

$$W(\Delta) = \sum_{c=1}^{\infty} W_c(\Delta), \quad (5.42)$$

coming from  $\mathbb{Z}_c$  orbifolds of  $AdS_2$ . Our objective then is to compute each of these terms exactly using localization. We compute the leading term  $W_1(\Delta)$  in §5.4 and the subleading terms corresponding to  $c > 1$  in §5.4.4.

## 5.2 Localization of Functional Integral in Supergravity

Evaluating the formal functional integral (3.14) over string fields for  $W(q, p)$  is of course highly nontrivial. To proceed further, we first integrate out the infinite tower of massive string modes and massive Kaluza-Klein modes to obtain a *local* Wilsonian effective action for the massless supergravity fields keeping all higher derivative terms. We can regard the ultraviolet finite string theory as providing a supersymmetric and consistent cutoff at the string scale. Our task is then reduced to evaluating a functional integral in supergravity. The near horizon geometry preserves eight superconformal symmetries and the action, measure, operator insertion, boundary conditions of the functional integral (3.14) are all supersymmetric<sup>12</sup>. The formal

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<sup>12</sup>Supersymmetry of the Wilson line and the action is discussed in the appendix.



supersymmetry of the functional integral makes it possible to apply localization techniques [17, 12] to evaluate it.

To apply localization to our system, we drop two gravitini multiplets to obtain a  $\mathcal{N} = 2$  theory and also drop the hypermultiplets to consider a reduced theory. This is partially motivated by the fact that the hypermultiplets are flat directions of the classical entropy function and our black hole is not charged under the gauge fields that belong to the gravitini multiplets. This theory contains a supergravity multiplet coupled to eight vector multiplets with a duality group

$$SO(6, 2; \mathbb{Z}) \times SL(2, \mathbb{Z}). \quad (5.43)$$

In the effective action for these fields we will further ignore the D-type terms. This is partially justified by the fact that the black hole horizon is supersymmetric and a large class of D-terms are known not to contribute to the Wald entropy as a consequence of this supersymmetry [105]. We will denote the functional integral (3.14) restricted to this reduced theory by  $\widehat{W}(q, p)$  which is what we compute in the subsequent sections. We find that  $\widehat{W}(q, p)$  itself agrees perfectly with (5.33) for  $d(q, p)$ . This rather nontrivial agreement can be regarded as post-facto evidence that the reduced theory correctly captures the relevant physics.

### 5.2.1 Functional Integral in $\mathcal{N} = 2$ Off-shell Supergravity

The renormalized action  $S_{ren}(\phi)$  (4.80) has the same functional form as the classical entropy function. In particular, its extrema  $\phi = \phi_*$  correspond to the attractor values of the scalar fields and its value at the extremum  $S_{ren}(\phi^*)$  equals the Wald entropy for the local Lagrangian described with a prepotential  $\mathcal{F}$ . However, the physics behind the renormalized action is completely different. Unlike the classical entropy function which is essentially a classical on-shell object, the renormalized action is a quantum object obtained after a complicated holographic renormalization procedure using an off-shell localizing field configuration (4.64). Even though the scalar fields in the localizing solution asymptote to the attractor values at the boundary of the  $AdS_2$ , they have a nontrivial coordinate dependence in the bulk and they take the value  $X_*^I + C^I$  at the center of  $AdS_2$ . In particular, they are excited away from their attractor values and are no longer at the minimum of  $S_{ren}$ . Even though the scalar fields thus ‘climb up the potential’ away from the minimum of the entropy function, the localizing solution remains Q-supersymmetric (in the Euclidean theory) because the auxiliary fields  $Y_{ij}^I$  get excited appropriately to satisfy the Killing spinor equations. This is what enables us to integrate over  $\phi$  for values in field space far away from the on-shell values.

The infinite dimensional functional integral (3.14) for the Wilson line in the reduced theory can thus be written as a finite integral (4.83)

$$\widehat{W}(q, p) = \int_{\mathcal{M}_Q} e^{-\pi\phi^I q_I} e^{\mathcal{F}(\phi, p)} |Z_{inst}|^2 Z_{det} [d\phi]_\mu \quad (5.44)$$

The measure of integration  $[d\phi]_\mu$  is computable from the original measure  $\mu$  of the functional integral of massless fields of string theory by standard collective coordinate methods. The factor  $Z_{det}$  is the one-loop determinant of the quadratic fluctuation operator around the localizing

instanton solution. Such one-loop determinant factors in closely related problems have been computed in [11, 99]. We have included  $|Z_{inst}|^2$  to include possible contributions from brane instantons which is partially captured by the topological string for a class of branes.

Note that the exponential of the integrand is in the spirit of the conjecture by Ooguri, Strominger, and Vafa [18]. Our treatment differs from [18] in that the natural ensemble in our analysis is the microcanonical one. Moreover, we will be able to determine the measure factor from first principles and to determine the subleading orbifolded localizing instantons that contribute to the functional integral. For earlier related work see [114, 44].

To compute  $\widehat{W}(q, p)$ , it is necessary to evaluate all these factors explicitly, as well as to perform the finite dimensional integral over  $\phi$ . This is what we will do for our system in §5.4. For the  $\mathcal{N} = 2$  reduction of the  $\mathcal{N} = 8$  theory that we consider,  $n_v = 7$  and the prepotential is given by

$$F(X) = -\frac{1}{2} \frac{X^1 C_{ab} X^a X^b}{X^0}, \quad a, b = 2, \dots, 7. \quad (5.45)$$

where  $C_{ab}$  is the intersection matrix of the six 2-cycles of  $T^4$ . In our normalization, it is given by

$$C_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{3 \times 3} \quad (5.46)$$

where  $\mathbf{1}_{3 \times 3}$  is a  $3 \times 3$  identity matrix. This prepotential describes the classical two-derivative supergravity action. Note that this does not depend on the field  $\hat{A}$  because there are no higher-derivative quantum corrections to the prepotential.

### 5.3 Integration measure

The measure  $[d\phi]_\mu$  is inherited from the standard measure on field space in the original functional integral. The collective coordinates  $\{\phi^I\}$  of the localizing instanton solutions correspond to the values of the scalar fields  $\{X^I\}$  at the center of the  $AdS_2$ . The functional integration measure for the scalar fields is a pointwise product of integration measure over the scalar manifold. The metric and hence the measure on the scalar manifold can be read off from the kinetic term of the scalar fields [87, 90]. The scalar kinetic action is

$$8\pi\mathcal{L} = \sqrt{|g|}g^{\mu\nu} \left[ i(\partial_\mu F_I + i\mathcal{A}_\mu F_I)(\partial^\mu \bar{X}^I - i\mathcal{A}^\mu \bar{X}^I) + h.c. \right], \quad (5.47)$$

where  $\mathcal{A}_\mu$  is the gauge field for the  $U(1)$  gauge symmetry of the off-shell supergravity theory. This field does not have a kinetic term and it is therefore determined by its equation of motion to be

$$\mathcal{A}_\mu^* = \frac{1}{2} \frac{\bar{F}_I \vec{\partial}_\mu X^I - \bar{X}^I \vec{\partial}_\mu F_I}{-i(\bar{F}_I X^I - F_I \bar{X}^I)}. \quad (5.48)$$

The Lagrangian  $8\pi\mathcal{L}^*$  computed by substituting  $\mathcal{A}_\mu^*$  in (5.47) becomes

$$-\sqrt{|g|}g^{\mu\nu} \left[ N_{IJ} \partial_\mu X^I \partial_\nu \bar{X}^J - \frac{e^{-K}}{4} (K_I \partial_\mu X^I - \bar{K}_I \partial_\mu \bar{X}^I)(K_I \partial_\nu X^I - \bar{K}_I \partial_\nu \bar{X}^I) \right], \quad (5.49)$$

with

$$N_{IJ} := -i(F_{IJ} - \bar{F}_{IJ}) = 2 \operatorname{Im}(F_{IJ}), \quad (5.50)$$

$$e^{-K} := -i(X^I \bar{F}_I - \bar{X}^I F_I), \quad (5.51)$$

$$K_I := \frac{\partial K}{\partial X^I} = ie^K (\bar{F}_I - F_{IJ} \bar{X}^J). \quad (5.52)$$

The metric  $g_{\mu\nu}$  is not the physical metric of Poincaré supergravity because it does not come with the canonical kinetic term. It is related to the dilatation-invariant physical metric  $G$  as

$$G_{\mu\nu} = e^{-K} g_{\mu\nu}, \quad (5.53)$$

whose kinetic term is given by the standard Einstein-Hilbert action. We have

$$\sqrt{|g|} g^{\mu\nu} = e^K \sqrt{|G|} G^{\mu\nu}. \quad (5.54)$$

It is natural to define the scalar functional integral measure using the physical metric  $G_{\mu\nu}$ . The measure can be determined by the metric induced by the inner product in field space:

$$(\delta X, \delta X) = \int d^4x \sqrt{|G|} \delta X \delta X. \quad (5.55)$$

Substituting  $X^I = (\phi^I + ip^I)/2$  in (5.49), and using (5.53), (5.54), we obtain the induced metric on the localizing submanifold in the field space

$$d\Sigma^2 = M_{IJ} \delta\phi^I \delta\phi^J, \quad (5.56)$$

with

$$M_{IJ} = e^K \left[ N_{IJ} - \frac{e^K}{4} (K_I - \bar{K}_I)(K_J - \bar{K}_J) \right]. \quad (5.57)$$

It is possible to write the metric on the localizing manifold entirely in terms of the Kähler potential<sup>13</sup>  $K$  (5.51). It is easy to check that

$$N_{IJ} = \frac{\partial^2 e^{-K}}{\partial X^I \partial \bar{X}^J} = e^{-K} \left( \frac{\partial^2 K}{\partial X^I \partial \bar{X}^J} - \frac{\partial K}{\partial X^I} \frac{\partial K}{\partial \bar{X}^J} \right). \quad (5.58)$$

Defining the metric  $K_{IJ}$  in terms of the Kähler potential in the usual way

$$K_{IJ} := \frac{\partial^2 K}{\partial X^I \partial \bar{X}^J}, \quad (5.59)$$

and using (5.58), we can write the Lagrangian (5.49) entirely in terms of the Kähler potential:

$$8\pi\mathcal{L} = -\sqrt{|g|} g^{\mu\nu} e^{-K} \left[ K_{IJ} \partial_\mu X^I \partial_\nu \bar{X}^J - \frac{1}{4} \partial_\mu K \partial_\nu K \right]. \quad (5.60)$$

---

<sup>13</sup>Upon gauge-fixing, on the space of projective coordinates  $K_{IJ}$  becomes the Kähler metric. We will refer to  $K$  as the Kähler potential even though we do not fix any gauge here.

Substituting  $X^I = (\phi^I + ip^I)/2$  in (5.60), we can rewrite the moduli space metric (5.56) as

$$M_{IJ} = K_{IJ} - \frac{1}{4} \frac{\partial K}{\partial \phi^I} \frac{\partial K}{\partial \phi^J}. \quad (5.61)$$

Since the metric  $K_{IJ}$  given in terms of the Kähler potential (5.59), this expresses the moduli space metric  $M_{IJ}$  entirely in terms of the Kähler potential. The measure on the localizing manifold is simply the measure induced by this metric and is given by

$$\prod_{I=0}^{n_v} d\phi^I \sqrt{\det(M)}. \quad (5.62)$$

## 5.4 Macroscopic Quantum Partition Function

The two-derivative action of  $\mathcal{N} = 8$  is invariant under the continuous duality group  $E_{7,7}(\mathbb{R})$ . We therefore expect to be able to write the macroscopic answer in terms of  $\Delta$  which is the unique quartic invariant of  $E_{7,7}(\mathbb{R})$ . For this purpose, we will first write the renormalized action in new variables so that it depends only on the invariant  $\Delta$  and then work out the measure in the same variables to obtain a manifestly duality invariant expression for the Wilson line.

### 5.4.1 Renormalized action and duality invariant variables

As discussed in §5.1.1 the electric and magnetic charge vectors  $Q$  and  $P$  respectively are related to the charges in the Type-IIA frame (5.8) by

$$Q = (q_0, -p^1; q_a) \quad P = (q_1, p^0; p^a) \quad . \quad (5.63)$$

The inner product is defined for example by

$$P \cdot P = 2 q^1 p^0 + p^a C_{ab} p^b, \quad (5.64)$$

The charge configuration (5.10) has only five nonzero charges  $q_0 = n$ ,  $q_1 = l$ ,  $p^1 = -w$ , and  $p^2$ ,  $p^3$ . Hence, the three T-duality invariants all have nonzero values given by

$$Q^2 = 2nw, \quad P^2 = 2p^2p^3, \quad Q \cdot P = wl. \quad (5.65)$$

The natural variables to start with are the projective coordinates

$$S := X^1/X^0, \quad T^a := X^a/X^0 \quad a = 2, \dots, n_v, \quad (5.66)$$

with real and imaginary parts defined by

$$S := a + is, \quad T^a := t^a + ir^a. \quad (5.67)$$

For our localizing instanton solutions we obtain

$$a = \phi^1/\phi^0, \quad s = -w/\phi^0 \quad (5.68)$$

$$t^a = \phi^a/\phi^0, \quad r^a = p^a/\phi^0. \quad (5.69)$$

The renormalized action (4.78) for this charge configuration and prepotential (5.45) is

$$S_{ren} = -\frac{\pi}{2\phi^0} [-w(\phi^2 - P^2) + 2\phi^1(\phi \cdot P)] - \pi n\phi^0 - \pi l\phi^1, \quad (5.70)$$

where  $\phi^2 = \phi^a C_{ab} \phi^b$  and  $\phi \cdot P = \phi^a C_{ab} P^b$ . Using the parametrization (5.66) and (5.67) and the T-duality invariants (5.65) it can be written as

$$S_{ren} = \frac{\pi}{2} \left[ P^2 s + \frac{Q^2}{s} + \frac{2Q \cdot P a}{s} \right] - \frac{\pi w^2 t^2}{2s} + \frac{\pi a w t \cdot P}{s}. \quad (5.71)$$

Our next goal will be to define integration variables to write the action entirely in terms of the U-duality invariant  $\Delta$ . Since the action is quadratic in the  $t^a$  variables, it is useful to complete the squares by defining

$$\tau^a = \frac{w}{\sqrt{s}} \left( t^a - \frac{a p^a}{w} \right) \quad (5.72)$$

so that

$$S_{ren} = \frac{\pi}{2} \left[ P^2 s + \frac{Q^2}{s} + \frac{P^2 a^2}{s} + \frac{2Q \cdot P a}{s} \right] - \frac{\pi \tau^2}{2}. \quad (5.73)$$

Note that the parenthesis is a manifestly S-duality invariant combination which is quadratic in the axion variable  $a$ . So we complete the square again by defining

$$\sigma = \frac{\pi P^2 s}{2}, \quad \alpha = \frac{1}{\sqrt{\sigma}} (P^2 a + Q \cdot P) \quad (5.74)$$

The renormalized action then becomes

$$S_{ren} = \left( \sigma + \frac{z^2}{4\sigma} \right) - \frac{\pi \tau^2}{2} + \frac{\pi \alpha^2}{2}. \quad (5.75)$$

with

$$z^2 = \pi^2 (Q^2 P^2 - (Q \cdot P)^2) = \pi^2 \Delta. \quad (5.76)$$

The variables  $(\sigma, \alpha, \tau^a)$  can be regarded as the duality invariant variables. We now turn to the integration measure.

#### 5.4.2 Conformal compensator, Gauge-fixing, and Analytic Continuation

The constants  $C^I$  which characterize the localizing instanton solution (4.64) are all real. Hence, the contour of integration for the variables  $s$  and  $t$  would appear to be along the real axis. The quadratic terms in  $t$  in the action (5.73) would lead to divergent Gaussian integrals. We will see below that this is nothing but the divergence of Euclidean quantum gravity arising from the integration over the conformal factor that has a wrong sign kinetic term.

We recall that the scalar kinetic term (5.60) can be written as

$$- \sqrt{-g} g^{\mu\nu} \left[ e^{-K} K_{IJ} \partial_\mu X^I \partial_\nu \bar{X}^J - \frac{1}{4} e^{-K} \partial_\mu K \partial_\nu K \right]. \quad (5.77)$$

The kinetic term for the spacetime metric  $g_{\mu\nu}$  is of the form<sup>14</sup>

$$-\frac{1}{6}\sqrt{-g}e^{-K}R_g, \quad (5.78)$$

We can thus identify  $e^{-K/2}$  as a conformal compensator  $\Omega$  which is often used to extend the gauge principle to include scale invariance in addition to diffeomorphism invariance. The Einstein-Hilbert action is then replaced by

$$\sqrt{-g} \left[ -\frac{1}{6}\Omega^2 R_g - g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega \right], \quad (5.79)$$

which is now invariant under both diffeomorphisms and Weyl rescalings. As can be seen from (5.77), the kinetic term for  $\Omega$  has a wrong sign compared to a physical scalar, as is usual for the conformal compensator field. In D-gauge [87]  $\Omega$  is gauge-fixed to a constant and one recovers the Einstein-Hilbert action. Our localizing solution is however in a different gauge in which the volume of  $AdS_2$  in the metric  $g$  is gauge-fixed and hence  $\Omega$  is effectively a fluctuating field. This also explains why we have  $n_v + 1$  scalar moduli  $\{\phi^I\}$  even though there are only  $n_v$  physical scalars. Essentially, our choice of gauge enables us to borrow the conformal factor  $\Omega$  as an additional scalar degree of freedom. The advantage is that the symplectic symmetry acts linearly on the fields  $\{\phi^I\}$ .

Since the kinetic term for conformal compensator  $\Omega$  has a wrong sign, to make the Euclidean functional integral well defined, it is necessary to analytically continue the contour of integration in field space [115]. For our prepotential (5.45), the Kähler potential is given by

$$\exp[-K] = 4 |X^0|^2 \text{Im}(S) C_{ab} \text{Im}(T^a) \text{Im}(T^b). \quad (5.80)$$

For  $S$  and  $T^a$  fixed, we see that  $\Omega$  is proportional to  $X^0$  up to a phase that can gauge-fixed by using the additional  $U(1)$  gauge symmetry. Thus, the analytic continuation in the  $\Omega$  space can be achieved by analytically continuing in the  $X^0$  space. For the localizing solution,  $X^0 = \phi^0$ . Thus, analytic continuation in  $\Omega$  space can be achieved by analytically continuing in the  $\phi^0$  space. Correspondingly, we take the contour of integration of  $\phi^0$  or equivalently of  $\sigma$  along the imaginary axis rather than along the real axis<sup>15</sup>.

A familiar example of such an analytic continuation is the functional integral for the worldsheet metric in first-quantized string theory. The conformal factor of the metric is the Liouville mode which can be thought of as a conformal compensator. Critical bosonic string with  $c = 26$  can be regarded as a noncritical string theory with  $c = 25$  coupled to this Liouville mode. The Liouville mode plays the role of time coordinate in target space [117] and has a wrong-sign kinetic term on the worldsheet. The corresponding functional integral then has to be defined by a similar analytic continuation [118].

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<sup>14</sup>We suppress an overall factor of  $1/8\pi$  that is irrelevant for the discussion here but is important for the normalization of the renormalized action in §5.4.

<sup>15</sup>In general there can be subtleties in such analytic continuation, see for example [116]. These will not be important in the present context.

### 5.4.3 Evaluation of the Localized Integral

The localizing action  $QV$  with abelian gauge fields is purely quadratic. Hence, the quadratic fluctuation operator around the localizing instantons does not depend on the collective coordinates  $\{C^I\}$ . As a result,  $Z_{det}$  is independent of  $\{\phi^I\}$  and charges can be absorbed in the overall normalization constant. Another simplification for the  $\mathcal{N} = 8$  theory is that  $|Z_{inst}|^2 = 1$  because the classical prepotential (5.45) that we have used is quantum exact.

Thus, all that remains is to compute the determinant of the matrix  $M_{IJ}$  introduced in (5.57). Since there are no terms that depend on  $\hat{A}$  for our prepotential, it is homogenous of degree 2 in the variables  $X$ . As a result,  $F_{IJ}X^J = F_I$ , and it follows from (5.52) that

$$K_I = e^K N_{IJ} \bar{X}^J, \quad \bar{K}_I = e^K N_{IJ} X^J. \quad (5.81)$$

This allows us to write (5.57) as

$$M_{IJ} = e^K \left( N_{IJ} + \frac{1}{4} e^K N_{IK} p^K N_{JL} p^L \right). \quad (5.82)$$

We have

$$\det(M) = \exp \left[ \frac{(n_v + 1)}{2} K \right] \det(N) \det(1 + \Lambda), \quad (5.83)$$

where the matrix  $\Lambda$  is defined by

$$\Lambda_J^I = \frac{1}{4} e^K p^I N_{JL} p^L. \quad (5.84)$$

Some elements of this measure such as the matrix  $N_{IJ}$  were anticipated in the work of [102, 103, 119] based on considerations of symplectic invariance. Our derivation follows from the analysis of the induced metric on the localizing manifold and has additional terms depending on  $K_I$  and  $\exp(K)$  which are also symplectic invariant. Unlike in the  $\mathcal{N} = 4$  theory, in the  $\mathcal{N} = 8$  theory the higher-derivative corrections are zero, and do not provide a useful guide for the determination of nonholomorphic terms of the measure such as the powers of  $\exp(K)$ .

It is easy to see that for our system  $\text{Tr}(\Lambda^n) = \lambda^n$  where  $\lambda$  is a numerical constant independent of charges. As a result,

$$\det(1 + \Lambda) = \exp(\text{Tr} \log(1 + \Lambda)) = \exp(\log(1 + \lambda)) \quad (5.85)$$

is a field-independent and charge-independent numerical constant. In what follows, we will ignore all such numerical constants in the evaluation of the measure and determine the overall normalization of the functional integral in the end.

Hence, up to a constant,  $\det(M)$  is determined by  $\det(N)$  and  $\exp(K)$ . For our prepotential, evaluating on the localizing instanton solution we obtain

$$\exp[-K] = 4 P^2 s = 8\sigma/\pi \quad (5.86)$$

which is manifestly duality invariant. Similarly,

$$\det(N) = \frac{s^{n_v-3} \det(C_{ab})}{4|X^0|^4} e^{-2K} = s^{n_v+3} \left(\frac{P^2}{w^2}\right)^2 \quad (5.87)$$

as can be checked using Mathematica. In terms of the duality invariant variables defined earlier, we see that the measure is given by

$$\prod_{I=0}^{n_v} d\phi^I \sqrt{\det(N)} = \frac{1}{\sqrt{\sigma}} d\sigma d\alpha \prod_2^{n_v} d\tau^a \quad (5.88)$$

up to an overall constant that is independent of charges and fields. The total measure is thus given by

$$\prod_{I=0}^{n_v} d\phi^I \sqrt{\det(M)} = \frac{d\sigma}{\sigma^{\rho+1}} d\alpha \prod_2^{n_v} d\tau^a \quad (5.89)$$

with  $\rho = n_v/2$ . Our total integral is hence manifestly duality invariant.

Performing the Gaussian integrals over  $\alpha$  and  $\tau^a$  we obtain

$$\int \frac{d\sigma}{\sigma^{\rho+1}} \exp\left(\sigma + \frac{z^2}{4\sigma}\right) \quad (5.90)$$

which gives exactly the integral representation of the Bessel function  $\tilde{I}_{7/2}(z)$  for  $n_v = 7$ . The overall numerical normalization needs to be fixed by hand but once it is fixed for one value of  $\Delta$ , one obtains a nontrivial a function for all other values of  $\Delta$  given by

$$W_1(\Delta) = \sqrt{2} \pi \left(\frac{\pi}{\Delta}\right)^{7/2} I_{7/2}(\pi\sqrt{\Delta}). \quad (5.91)$$

This macroscopic calculation thus precisely reproduces the first term with  $c = 1$  in (5.42) and matches beautifully with the first term in (5.39) from the Rademacher expansion (5.33) for of the microscopic degeneracy  $d(\Delta)$ .

For large  $z$ , the Bessel function has an expansion

$$I_\rho(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left[ 1 - \frac{(\mu-1)}{8z} + \frac{(\mu-1)(\mu-3^2)}{2!(8z)^3} - \frac{(\mu-1)(\mu-3^2)(\mu-5^2)}{3!(8z)^5} + \dots \right], \quad (5.92)$$

with  $\mu = 4\rho^2$ . The exponential term  $\exp(\pi\sqrt{\Delta})$  gives the Cardy formula and  $\pi\sqrt{\Delta}$  can be identified with the Wald entropy of the black hole. Higher terms in the series give power-law suppressed finite size corrections to the Wald entropy. This is however not a convergent expansion but only an asymptotic expansion. This means that for any given  $z$  only the first few terms of order some power of  $z$  are useful for making an accurate estimate. Beyond a certain number of terms, including more terms actually makes the estimate worse rather than improve it. For larger and larger  $z$  one can include more or more terms and one obtains better and better approximation but it is never convergent.



It should be emphasized that our computation of  $W_1(\Delta)$  gives an exact integral representation (5.90) of the Bessel function  $I_{7/2}(z)$  and not merely the asymptotic expansion (5.92). This is made possible because localization gives an exact evaluation of the functional integrals and allows one to access large regions in the field space far away from the classical saddle point of the entropy function used to derive the Cardy formula.

It is instructive to compare the integers  $d(\Delta)$  with the  $W_1(\Delta)$  and the exponential of the Wald entropy (2) we tabulate these coefficients for the first few values of  $\Delta$ .

**Table 2:** Comparison of the microscopic degeneracy  $d(\Delta)$  with the functional integral  $W_1(\Delta)$  and the exponential of the Wald entropy. The last three rows in the table equal each other asymptotically.

$\Delta$	-1	0	3	4	7	8	11	12	15
$d(\Delta)$	1	2	8	12	39	56	152	208	513
$W_1(\Delta)$	1.040	1.855	7.972	12.201	38.986	55.721	152.041	208.455	512.958
$\exp(\pi\sqrt{\Delta})$	-	1	230.765	535.492	4071.93	7228.35	33506	53252	192401

Note that the area of the horizon goes as  $4\pi\sqrt{\Delta}$  in Planck units. Already for  $\Delta = 12$  this area would be much larger than one, and one might expect that the Bekenstein-Hawking-Wald entropy would be a good approximation to the logarithm of the quantum degeneracy. However, we see from the table that these two differ quite substantially. Indeed, in this example, since there are no higher-derivative local terms, Wald entropy equals the Bekenstein-Hawking entropy. The discrepancy thus arises entirely from the quantum contributions from integrating over massless fields. Localization enables an exact evaluation of these quantum effects. The resulting  $W_1(\Delta)$  is in spectacular agreement with  $d(\Delta)$  and in fact comes very close to the actual integer even for small values of  $\Delta$ .

We see from the asymptotic expansion (5.92) that the subleading logarithmic correction to the Bekenstein-Hawking entropy goes as  $-2\log(\Delta)$ . This is in agreement with the results in [75, 77, 76] where the logarithmic correction was computed by evaluating one-loop determinants of various massless fields around the classical background. Using localization, this logarithmic correction follows essentially from the analysis of the induced measure on the localizing manifold without the need for any laborious evaluation of one-loop determinants. Moreover, since localization accesses regions in field space very off-shell from the classical background the entire series of power-law suppressed terms in (5.92) follow with equal ease.

#### 5.4.4 Nonperturbative Corrections, Orbifolds, and Localization

We have seen that localization correctly reproduces the first term in the Rademacher expansion. This term already captures all power-law and logarithmic corrections to the leading Bekenstein-Hawking-Wald entropy exactly to all orders. We turn next to the computation of the higher terms in the Rademacher expansion (5.33) with  $c > 1$ . These terms are nonperturbative because they are exponentially suppressed with respect to the terms in (5.92).

It was proposed in [54, 73, 120, 12] that such non-perturbative corrections could arise from  $\mathbb{Z}_c$  orbifolds for all positive integers  $c$  because such orbifolds respect the same boundary conditions (3.4) on the fields. In general, it is difficult to justify keeping such subleading exponentials if the power-law suppressed terms are evaluated only in an asymptotic expansion. However, localization gives an exact integral representation of the leading Bessel function in §5.4.3. The power-law suppressed contributions are computed exactly, and it is justified to systematically take into account the exponentially suppressed contributions.

The  $\mathbb{Z}_c$  orbifold configurations that contribute to the localization integral are obtained as follows. We mod out with a symmetry  $R_c T_c$  which combines a supersymmetric order  $c$  twist  $R_c$  on  $AdS_2 \times S^2$  with an order  $c$  shift  $T_c$  along the  $T^6$ . The orbifold twist is required to be supersymmetric because to preserve the  $Q$  supercharge used for localization, the orbifold action must commute with  $L - J$  [12]. At the center of  $AdS_2$  and at the poles of  $S^2$  the twist looks like a generator of the supersymmetric  $C^2/\mathbb{Z}_c$  orbifold. With an appropriate shift, this action is freely acting and can be used to get smooth solutions [120].

To illustrate how this works together with localization let us first discuss the case when  $T_c(\delta)$  is a simple shift of  $2\pi\delta/c$  along the circle  $S^1$ . It acts on the momentum modes by

$$T_c(\delta) |m\rangle = e^{\frac{2\pi i \delta m}{c}} |m\rangle. \quad (5.93)$$

Let  $\phi$  be the azimuthal angle along the  $S^2$  and  $y$  be the coordinate of the circle  $S^1$  with  $2\pi$  periodicities. We will denote the orbifolded coordinates with a tilde. The orbifold operator  $R_c T_c$  identifies points in  $AdS_2 \times S^2 \times S^1$  with the identification

$$(\tilde{\theta}, \tilde{\phi}, \tilde{y}) \equiv (\tilde{\theta} + \frac{2\pi}{c}, \tilde{\phi} - \frac{2\pi}{c}, \tilde{y} + \frac{2\pi\delta}{c}) \quad (5.94)$$

The combined action  $R_c T_c(\delta)$  means that as we go around the boundary of  $AdS_2$  the momentum modes pick up a phase as in (5.93). This corresponds to turning on a Wilson line of the Kaluza-Klein gauge field  $\mathcal{A}$  that couples to the momentum  $n$  by modifying the gauge field as

$$\mathcal{A} = -ie_*(\tilde{r} - 1)d\tilde{\theta} + \delta d\tilde{\theta} \quad (5.95)$$

The metric on the orbifolded  $AdS_2$  factor has the same form

$$ds^2 = v_* \left[ (\tilde{r}^2 - 1)d\tilde{\theta}^2 + \frac{d\tilde{r}^2}{(\tilde{r}^2 - 1)} \right] \quad 1 \leq \tilde{r} < \tilde{r}_0; \quad 0 \leq \tilde{\theta} < \frac{2\pi}{c} \quad (5.96)$$

as the original unorbifolded metric (4.10) but the  $\tilde{\theta}$  variable now has a different periodicity and we have cutoff at  $\tilde{r} = \tilde{r}_0$ . Thus, it is not immediately obvious that asymptotic conditions on the fields are the same as for the unorbifolded theory. To see this, we change coordinates

$$\tilde{\theta} = \frac{\theta}{c}, \quad \tilde{\phi} = \phi - \frac{\theta}{c}, \quad \tilde{y} = y + \frac{\theta}{c}, \quad \tilde{r} = cr, \quad (5.97)$$

so that in the new coordinates, the fields have the same asymptotics (3.4) as before:

$$ds_2^2 \sim v_* \left[ r^2 d\theta^2 + \frac{dr^2}{r^2} \right], \quad \mathcal{A} \sim -ie_* r d\theta. \quad (5.98)$$

Moreover, the new coordinates have the same identification

$$(\theta, \phi, y) \equiv (\theta + 2\pi, \phi, y) \equiv (\theta, \phi + 2\pi, y) \equiv (\theta, \phi, y + 2\pi) \quad (5.99)$$

as in the unorbifolded theory. Such orbifolded field configurations with the same asymptotic behavior will therefore contribute to the functional integral.

The orbifold action is freely acting if  $\delta$  and  $c$  are relatively prime. Therefore, the localizing equations, which are local differential equations, remain the same as before and one obtains the same localizing instantons (4.64) as before. To compute the renormalized action it is convenient to use the tilde coordinates. If we put a cutoff at  $r_0$ , the range of  $r$  is  $1/c \leq r \leq r_0$  and that of  $\tilde{r}$  is  $1 \leq \tilde{r} \leq cr_0$ . The physical action is an integral of the same local Lagrangian density as the unorbifolded theory but now the ranges of integration are different. Since the localizing instantons do not depend on the angular coordinates, the nontrivial integration is over the coordinate  $\tilde{r}$ . The  $r_0$  dependent contribution from this integral is therefore  $c$  times larger than before but the  $r_0$  independent constant piece is the same as before. On the other hand, from the angular integrations one gets an overall factor of  $1/c$  because the range of these coordinates is divided by  $c$  by the identification (5.94). Altogether, the renormalized action obtained by removing the  $r_0$  dependent divergence is smaller by a factor of  $c$ . Moreover, with the modified gauge field (5.95) the Wilson line contributes an additional phase. In summary, instead of (3.13) we obtain

$$\exp \left[ \frac{\mathcal{S}_{ren}(\phi)}{c} + \frac{2\pi i n \delta}{c} \right], \quad (5.100)$$

where  $\mathcal{S}_{ren}$  is the unorbifolded renormalized action for the localizing instantons given by (5.70).

Since the phase above does not depend on  $\phi$  we can integrate over  $\phi$  as before and then sum over all phases. Thus  $W_c$  factorizes as

$$W_c(\Delta) = A_c(\Delta) B_c(\Delta) \quad (5.101)$$

where  $A_c$  comes from integration over  $\phi$  and  $B_c$  comes from the sum over phases. Since the renormalized action is now smaller by a factor of  $c$ , it is easy to see that the integral  $A_c$  gives precisely the modified Bessel function but with an argument  $z_c = z/c$  with possible powers of  $c$  coming from the measure which we absorb for now in  $B_c(\Delta)$ . The final answer thus has the form

$$W_c(\Delta) = \sqrt{2} \pi \left( \frac{\pi}{\Delta} \right)^{7/2} I_{\frac{7}{2}} \left( \frac{\pi \sqrt{\Delta}}{c} \right) B_c(\Delta). \quad (5.102)$$

This is very close to the  $c$ -th term in the Rademacher expansion. To obtain agreement we would need to show

$$B_c(\Delta) = c^{-9/2} K_c(\Delta). \quad (5.103)$$

We see from (5.34) that the Kloosterman sum is also a rather intricate sum over various  $c$  and  $\Delta$  dependent phases. This suggests that by summing over the phases for various allowed

orbifolds and properly fixing their relative normalization with respect to the  $c = 1$  term, it may be possible to compute  $B_c(\Delta)$  to reproduce the desired expression (5.103) in terms of the Kloosterman sum [121].

## 6. Macroscopic index and $AdS_3/CFT_2$ correspondence

In this section we use an intermediate approach based on  $AdS_3$  rather than  $AdS_2$  to compute finite charge corrections to the black hole entropy. The computation is exact in the limit when only one of the charges is taken to be very large keeping the other charges finite. The answer is particularly sensitive to the details of the phase we are working on and therefore we can learn about microscopic details of the theory. In the process we find easier to construct a macroscopic index which captures all the degrees of freedom from horizon till asymptotic infinity. In the limit considered it matches precisely with a microscopic computation. We state our main result as: *in the limit when only one of the charges is taken to be very large, the asymptotic growth of the macroscopic index which has the form of a Cardy formula is controlled by an effective central charge which is related to the coefficients of the Chern-Simons terms computed at asymptotic infinity.*

In section §3, we explained that the index computed at asymptotic infinity captures not only the horizon degrees of freedom but also any exterior contribution that sits between  $AdS_2$  and the asymptotic infinity. The total index is a combination of an index for the horizon and other for the hair degrees of freedom. A construction of such an index from the gravity side is the subject of this section.

The idea is to construct the index starting from the horizon and going gradually to asymptotic infinity. As explained in the previous sections, to count the horizon degrees of freedom, we should perform a path integral over the string fields in  $AdS_2$  with specific boundary conditions. In general this is a very difficult problem even though we showed in the previous sections that localization allows to go very far. By including the contribution from the hair modes we will find easier to compute the index instead of the degeneracy.

For extremal black holes with a  $AdS_2 \times S^1$  factor in the near horizon geometry, we can use the power of  $AdS_3$  to study subleading corrections to the Beckenstein-Hawking entropy [35].

The near horizon geometry  $AdS_2 \times S^1$  has isometry group  $SL(2, \mathbb{R}) \times U(1)$  while  $AdS_3$  has  $SO(2, 2) = SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ . Once we make the radius of  $S^1$  larger and larger we can restore the  $SO(2, 2)$  isometry and view the solution as a BTZ black hole living in  $AdS_3$ . Using the  $AdS_3/CFT_2$  correspondence we compute the entropy using a Cardy formula. The central charges of the dual  $CFT_2$  can be computed exactly by knowing the coefficients of the Chern-Simons terms on  $AdS_3$  [38].

This section is organized as follows. In section §6.1 we explain the criteria for which the index equals degeneracy in the large charge limit and how we can use the power of  $AdS_3$  to compute the quantum corrected entropy. In §6.2 we consider a puzzle of M5-branes on  $K3 \times T^2$  and its relation to four dimensional black holes. In §6.3 we consider the contribution of exterior modes to the asymptotics of the macroscopic index. In §6.5 we derive the asymptotic behaviour

of the index from microscopics. We end presenting various results for five dimensional black holes.

### 6.1 From $AdS_2$ to $AdS_3$

To determine  $B_6$  by computing first  $B_{hor}$  and then  $B_{hair}$  as defined in formula (3.19) is a very difficult task. First it requires evaluating a string path integral on  $AdS_2$  and second it requires determining the hair modes which is not an easy task [69, 70]. For these reasons we shall give an alternative approach based on  $AdS_3$  rather than on  $AdS_2$ . The main advantage, as we will see, resides on the fact that it is possible to determine exactly the central charges by just knowing the Chern-Simons terms in the bulk of  $AdS_3$ . Further we will see that the hair analyses simplifies once we combine them with the bulk contribution.

Consider a black hole whose near horizon geometry contains a factor  $AdS_2 \times S^1$  with metric

$$ds^2 = v \left( -r^2 dt^2 + \frac{dr^2}{r^2} \right) + \left( dy - \frac{r}{R} dt \right)^2 \quad (6.1)$$

The coordinate  $y$  corresponds to the circle and  $R$  is its radius. The fiber  $r/R dt$  corresponds, from the two dimensional point of view, to a charged electric gauge field. In particular if  $v = 1/R^2$  the space is locally  $AdS_3$ .

The space  $AdS_2 \times S^1$  has isometry group  $SL(2, \mathbb{R}) \times U(1)$ . The first factor corresponds to  $AdS_2$  isometries while the  $U(1)$  corresponds to translations along the circle. It differs from global  $AdS_3$  due to a translation identification of  $2\pi$  along the non-compact  $y$  direction. If we make the radius  $R$  very large we can restore the  $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$  of  $AdS_3$ . At the same time we take the asymptotic value of  $R$  to be very large keeping all the other moduli fixed to ensure that the space develops an intermediate  $AdS_3$  factor. Now we can regard the black hole solution as an extremal BTZ black hole living in this intermediate  $AdS_3$  space time [122, 123]. For an extremal BTZ black hole, in the limit of very large spin  $J$ , we can use a Cardy formula to compute the entropy [124, 38, 125],

$$d(n \gg 1) \approx e^{2\pi \sqrt{nc_L/6}} \quad (6.2)$$

From the  $AdS_2$  point of view, the spin  $J$  is identified with the momentum  $n$  along the circle  $S^1$  which justifies the limit  $n \rightarrow \infty$ , that is the Cardy limit.

Since we expect the Cardy formula to hold in the full quantum theory we should take this as the quantum generalization of the black hole entropy. The problem is reduced to the computation of the central charges. Since we don't know the details of the dual  $CFT_2$ , we should be able to compute them from the bulk theory after including quantum corrections.

There are however some subtleties in this approach. The Cardy formula should count the total degeneracy without caring if it is a single or multi center solution in  $AdS_3$ . It should include not just the horizon degrees of freedom but also any possible contribution of additional modes exterior to  $AdS_2$  but living inside  $AdS_3$ . This should not be a problem because at the end we want to compare the bulk answer with a microscopic index which already includes all those

contributions. Moreover we neglect the possibility of multiple  $AdS_3$  throat [122] by working in an appropriate domain in the moduli space.

One additional problem we face is related to the fact that the  $CFT_2$  does not capture all the degrees of freedom of the black hole. There could be additional modes living on the boundary, like in the case of the  $U(1)$  factor in  $AdS_5$  [126], or between  $AdS_3$  and asymptotic infinity. We shall call them exterior modes.

Since we take the asymptotic value of the radius to infinity while keeping its attractor value large but fixed the physical momentum, measured at infinity, vanishes. This restores the  $1+1$  Lorentz symmetry of part of the solution lying between  $AdS_3$  and asymptotic infinity. Therefore we expect the dynamics of the exterior modes to be described by a two dimensional field theory. In addition we have to combine this contribution with the Cardy formula to recover the total index.

We repeat the computation (3.19) but now rewriting the index in terms of the bulk  $AdS_3$  and exterior degrees of freedom,

$$B_6 = -\frac{1}{6!} \text{Tr}(-1)^{2h_{bulk}+2h_{exterior}} (2h_{bulk} + 2h_{exterior})^6 \quad (6.3)$$

$$= -\frac{1}{6!} \text{Tr}(-1)^{2h_{bulk}} \text{Tr}(-1)^{2h_{exterior}} (2h_{exterior})^6 \quad (6.4)$$

$$= \sum_{q+\tilde{q}=Q} B_{bulk}(q) B_{6_{exterior}}(\tilde{q}) \quad (6.5)$$

where we defined  $B_{bulk} = \text{Tr}(-1)^{2h_{bulk}}$  and  $B_{6_{exterior}} = -\frac{1}{6!} \text{Tr}(-1)^{2h_{exterior}} (2h_{exterior})^6$ . We have assumed that the black hole when regarded as a solution in  $AdS_3$  doesn't break any further supersymmetry. Under this assumption the fermionic zero modes are all part of the exterior modes which implies that only the term  $(2h_{exterior})^6$  will survive in the binomial expansion of  $(2h_{bulk} + 2h_{exterior})^6$ .

We should note that in the BTZ near horizon geometry  $AdS_2 \times S^1$  the time circle is contractible in the full  $AdS_3$  geometry while the  $S^1$  circle is not. On the boundary theory we should compute a partition function with anti periodic conditions along the thermal circle and periodic along  $S^1$ . In other words, the holographic correspondence instructs us to compute  $\text{Tr}(1)$  with Ramond-Ramond boundary conditions instead of  $\text{Tr}(-1)^F$  as presupposed by  $B_{bulk}$ . Instead we will try to argue that the asymptotics of left moving excitations are not changed after the insertion of  $(-1)^F$  in the trace. That is, for large momentum  $n$   $B_{bulk}$  is still given by a Cardy formula

$$B_{bulk} \approx e^{\pi \sqrt{n c_L/6}} \quad (6.6)$$

where  $c_L$  is the central charge of the left-Virasoro algebra of the  $CFT_2$ .

We find useful to construct the partition function

$$\hat{B}_6(\vec{q}, \tau) = \sum_n B_6(\vec{q}, n) e^{2\pi i n \tau} \quad (6.7)$$

$$= \hat{B}_{bulk}(\vec{q}, \tau) \hat{B}_{6_{exterior}}(\tau) \quad (6.8)$$

where  $\vec{q}$  stands for a generic charge vector and  $n$  is the momentum along the circle  $S^1$ . We are assuming that the exterior modes don't carry any other charge rather than momentum  $n$ .

The Cardy formula gives the asymptotic behavior of the degeneracy  $d = \text{Tr}(1)$  for large  $n$ . The derivation is based on the fact that under a modular transformation we can relate small with large  $\tau$  behavior. The partition function can then be related to its ground state energy via the formula

$$\hat{d}(\tau) \approx e^{\frac{\pi i c_L}{12\tau}} \quad (6.9)$$

with  $-c_L/24$  the ground state energy of the left-moving sector. Now instead of computing  $\hat{d}(\tau)$  we should compute  $\hat{B}_{bulk}(\tau)$  which requires an insertion  $(-1)^{2h_{bulk}}$  in the trace. Generically for a black hole that preserves at least four supercharges the isometries of  $AdS_3$  together with supersymmetries give rise to a  $(0, 4)$  superconformal algebra in the boundary theory. This includes an  $SU(2)$  R-symmetry current whose global part can be identified with the rotation symmetry group  $SU(2)$  of the black hole. Namely,  $h_{bulk}$  is identified with the zero mode of the  $U(1)$  in the  $SU(2)$  R-current algebra in the  $SCFT_2$ . Since the twist  $(-1)^{2h_{bulk}}$  by the zero mode of the right-moving current is not expected to change the ground state energy of the left-moving sector, we conclude that, for small  $\tau$ ,  $\hat{B}(\tau)$  still has the same behavior as  $\hat{d}(\tau)$ , that is,

$$\hat{B}(\tau) \approx e^{\frac{\pi i c_L}{12\tau}}, \quad |\tau| \ll 1 \quad (6.10)$$

The analysis for  $\hat{B}_{exterior}$  is a bit different. We will try to argue in the next section that the asymptotic growth of the index  $\hat{B}_{6exterior}$  is still given by a Cardy formula, but in this case is not possible to identify  $c_L$  with the central charge of a dual conformal field theory. We will be able to show that, under certain assumptions,  $\hat{B}_{exterior}(\tau)$  has a behavior given by (6.10)

$$\hat{B}_{exterior} \approx e^{\frac{\pi i c_L^{eff}}{12\tau}}, \quad |\tau| \ll 1 \quad (6.11)$$

with  $c_L^{eff}$  a constant that effectively controls the asymptotic growth. This should not be confused with the physical central charge.

A simple calculation shows that, in the large  $n$  limit, the growth of the total index is given by a Cardy like formula

$$B_6 = \oint \hat{B}_{bulk}(\tau) \hat{B}_{exterior}(\tau) e^{-2\pi i n \tau} \quad (6.12)$$

$$\approx e^{2\pi \sqrt{nc_L^{total}/6}}, \quad (6.13)$$

with  $c_L^{total} = c_L + c_L^{eff}$ .

In [38] the authors show, using general properties of  $AdS/CFT$  correspondence, that the left and right central charges  $c_L$  and  $c_R$  of the  $CFT_2$  can be computed from the bulk lagrangian via the coefficients of the Chern-Simons terms.

The gravitational Chern-Simons term constructed out of the gravitational  $SO(2, 1)$   $\Gamma$  connection

$$\Omega_3(\Gamma) = \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \quad (6.14)$$

induces a diffeomorphism anomaly which translates in the non-conservation of the boundary stress tensor. Holography relates the coefficient of  $\Omega_3$  to  $c_L - c_R$ , that is, the difference between the left and right-*Virasoro* central charges. In addition there can be gauge Chern-Simons terms of the form

$$\Omega_3(A) = A \wedge dA + \frac{2}{3} A \wedge A \wedge A \quad (6.15)$$

where  $A$  is a non-abelian gauge field. Similarly, the lack of gauge invariance induces a non-conserved current in the boundary theory. In the case of the  $SU(2)_L \times SU(2)_R$  R-symmetry, the coefficient of the Chern-Simons of each of the  $SU(2)$  factors is related to the R-current anomalies  $k_L$  and  $k_R$  respectively. These results are exact in the sense that they include already all the higher derivative corrections.

A “classical” approach to this problem consists in computing the central charges as a perturbative expansion in inverse powers of  $l^2$ , the size of  $AdS_3$ . This is called c-extremization [38] and requires, as the name suggests, extremizing the bulk action with respect to the parameter  $l$ .

Consider the example of string theory on  $AdS_3 \times S^p$ . We take the metric to be

$$ds^2 = l^2 (d\eta^2 + \sinh^2 \eta d\Omega_2^2) + l_{S^p}^2 d\Omega_p^2. \quad (6.16)$$

The values of  $l$  and  $l_{S^p}$  can be obtained by demanding the lagrangian computed on the background constant solution to be stationary under variation of the those parameters as implied by the equations of motion. In general the lagrangian  $\mathcal{L}_{p+3}$  is a very complicated function of the size  $l$  and the charges. Its dependence will come for example from non trivial fluxes on  $S^p$  or higher derivative corrections. Due to the topological nature of the Chern-Simons terms these will not be relevant for c-extremization.

Now, put the  $CFT_2$  in the boundary sphere  $S^2$  which has metric

$$ds^2 = e^{2w} d\Omega_2^2. \quad (6.17)$$

and consider a small variation  $\delta w$  in the free energy  $F$  defined via the partition function  $Z = e^{-F}$ ,

$$\delta F = \frac{1}{4\pi} \int d^2x \sqrt{g} T^{ij} \delta g_{ij} = \frac{\delta w}{2\pi} \int d^2x \sqrt{g} T^{ij} g_{ij} = \frac{\delta w}{2\pi} \int d^2x \sqrt{g} T_i^i \quad (6.18)$$

Using the trace anomaly equation

$$T_i^i = -\frac{c}{12} R \quad (6.19)$$

with  $c$  the central charge and  $R$  the Ricci scalar, we find

$$\frac{\delta F}{\delta w} = -\frac{c}{3} R. \quad (6.20)$$

To compute  $F$  from the bulk lagrangian we take the vacuum background  $AdS_3$  with some possible fluxes on  $S^p$  corresponding to the charges. The lagrangian becomes a constant scalar that depends on  $l_{S^p}$ ,  $l$  and the other charges

$$F = V_{S^2} V_{S^p} \mathcal{L}_{p+3} \int d\eta \sinh^2 \eta + S_{bdy} \quad (6.21)$$



The counter-term  $S_{bndy}$  is introduced to remove the infrared divergence associated with the infinite volume of  $AdS_3$ . This is all analogous to the discussion of the  $AdS_2$  path integral in the context of the quantum entropy function. For a large cutoff  $\eta_{max}$ , equation (6.21) becomes

$$F = V_{S^2} V_{S^p} \mathcal{L}_{p+3} \left( -\frac{1}{2} \eta_{max} + \frac{1}{2} e^{2\eta_{max}} \right) + S_{bndy} \quad (6.22)$$

The counter term  $S_{bndy}$  goes as  $e^{2\eta_{max}}$ . From here we also conclude that  $\omega = \eta_{max}$ . If we choose an appropriate boundary counter term we can eliminate the divergent term in (6.22) proportional to  $e^{2\eta_{max}}$ . We are left with

$$F = -\frac{1}{2} \eta_{max} V_{S^2} V_{S^p} \mathcal{L}_{p+3} \quad (6.23)$$

which is divergent since it still depends linearly on the cutoff. However because we are only interested in variations of  $F$  with respect to  $w$  or  $\eta_{max}$  the final result is finite

$$\frac{\delta F}{\delta w} = -\frac{1}{2} V_{S^2} V_{S^p} \mathcal{L}_{p+3} \quad (6.24)$$

Using (6.24) and (6.20) we compute the central charge

$$c = \frac{3}{2} V_{S^2} V_{S^p} \mathcal{L}_{p+3}. \quad (6.25)$$

This formalism is very similar to that of the entropy function [7, 68]. There the entropy was given by extremizing some entropy function proportional to the bulk lagrangian. Here we extremize a bulk lagrangian to compute the central charge.

For the moment we considered the case when both the left and right central charges are the same. Though this method is very powerful it requires knowing all the terms in the action which are non zero in the background solution. In the following we show how to go beyond this perturbative approach by considering the Chern-Simons terms.

The presence of gravitational Chern-Simons terms renders the bulk action diffeomorphic anomalous. This translates in the non conservation of the boundary stress tensor. Similarly the presence of other gauge Chern-Simons terms implies the non-conservation of the dual currents in the boundary theory [36, 38, 37].

The  $AdS/CFT$  correspondence instructs to compute a path integral over string fields with specific boundary conditions [36]

$$Z_{string} = e^{-S_{string}^{cl} - S_{string}^{1-loop} + \dots} = e^{-\Gamma[g_{ij}^0, A_0]} \quad (6.26)$$

where  $A_0$  is the boundary value of the gauge field and  $g_{ij}^0$  the boundary metric, just as an example. The effective action  $\Gamma[g_{ij}^0, A_0]$  includes both classical and quantum contributions. The  $AdS/CFT$  dictionary then tells us to compute the boundary correlation functions of the dual operators via

$$e^{-\Gamma[g^0, A]}|_{AdS_3} = \langle e^{-\int d^2x T^{ij} g_{ij}^0 - \int d^2x \text{Tr} A_\mu(x) J^\mu(x)} \rangle|_{CFT_2} \quad (6.27)$$

where  $T^{ij}$  is the boundary stress tensor and  $J^\mu$  is the dual current. The presence of a gauge Chern-Simons renders  $\Gamma[A]$  not gauge invariant and consequently it induces an anomalous dual current. A gauge transformation with parameter  $\Lambda$  gives

$$\delta_\Lambda \Gamma[A] = - \int d^2x \text{Tr} \Lambda J^\mu{}_{,\mu}(x) \quad (6.28)$$

where  $\Lambda$  denotes a gauge transformation. Similarly a gravitational Chern-Simons will render the boundary stress tensor anomalously conserved.

For four dimensional black holes which preserve at least four supercharges the near horizon geometry is guaranteed to be rotational symmetric as explained in section §3. Apart from an  $S^2$  factor the near horizon geometry is  $AdS_2 \times S^1$ . When we take the momentum and the asymptotic radius of  $S^1$  to infinity we can view the black hole as an extremal BTZ living in  $AdS_3$ . In this case the bulk theory has  $(0, 4)$  supersymmetry in  $AdS_3$ . Generalizations to  $(4, 4)$  theories corresponding to string theory on  $AdS_3 \times S^3$  are straightforward. We will describe the case of five dimensional black holes by the end of this section.

In the case of  $(0, 4)$  supersymmetry, the supergravity action can be regarded as a gauge theory with supergroup  $SU(1, 1) \times SU(1, 1|2)$ . In terms of the superconnections  $\Gamma_L$  and  $\Gamma_R$ , in the adjoint representation of  $SU(1, 1)$  and  $SU(1, 1|2)$  respectively, the action can be written as [68]

$$S = a_L \int \text{Tr}(\Gamma_L \wedge d\Gamma_L + \frac{2}{3}\Gamma_L \wedge \Gamma_L \wedge \Gamma_L) + a_R \int \text{Tr}(\Gamma_R \wedge d\Gamma_R + \frac{2}{3}\Gamma_R \wedge \Gamma_R \wedge \Gamma_R). \quad (6.29)$$

In terms of the bosonic fields the action is

$$\begin{aligned} S = & \int d^3x \sqrt{g}(R - 2\Lambda) \\ & + \frac{K}{2} \int \text{Tr} \left( \Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma \right) + \frac{k_R}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3}A \wedge A \wedge A \right) \end{aligned} \quad (6.30)$$

where  $\Lambda = -1/l^2$  is the cosmological constant,  $A$  is the  $SU(2)$  gauge field and  $\Gamma$  is the  $SO(2, 1)$  tangent space connection.

Under a local Lorentz gauge transformation  $\omega$  of  $\Gamma$ ,

$$\delta\Gamma = d\omega + [\Gamma, \omega] \quad (6.31)$$

the variation of the gravitational Chern-Simons gives a boundary term which induces a variation on the action

$$\delta S = \frac{K}{2} \int_{\partial AdS_3} \text{Tr} \omega d\Gamma. \quad (6.32)$$

At the same time a variation is induced in the source that couples to the stress tensor  $T^{ij}$  in the boundary,

$$\delta S_{CFT} = \frac{1}{2} \int d^2x \sqrt{g} T^{ij} \delta g_{ij} \quad (6.33)$$

The local Lorentz gauge transformation  $\omega$  induces a variation in the boundary metric. After some tedious calculation we can relate the difference between the left and right central charges of the CFT to the coefficient of the gravitational Chern-Simons term

$$c_L - c_R = 48\pi K \quad (6.34)$$

with  $K/2$  the coefficient of the gravitational Chern-Simons.

A similar analysis can be done for the gauge Chern-Simons. In this case the coefficient of the  $SU(2)$  gauge Chern-Simons is related to the anomaly  $k_R$  of the boundary  $SU(2)$  R-current via

$$k_R = 4\pi\alpha \quad (6.35)$$

where  $\alpha$  is the coefficient of the gauge Chern-Simons. Closure of the  $(0, 4)$  superconformal symmetry in the boundary CFT relates the central charge  $c_R$  of the right super-Virasoro algebra to the  $SU(2)$  R-current anomaly via

$$c_R = 6k_R$$

Armed with both the gravitational and gauge Chern-Simons coefficients  $\beta$  and  $\alpha$  we can compute  $c_L$

$$c_L = 96\pi\beta + 24\pi\alpha \quad (6.36)$$

In the case of string theory on  $AdS_3 \times S^3$  the R-symmetry is  $SO(4) = SU(2)_L \times SU(2)_R$ , the isometry group of  $S^3$ . In this case we have an additional gauge Chern-Simons associated with the  $SU(2)_L$  connection. In addition if the SCFT has  $(4, 4)$  superconformal symmetry we can identify the left central charge with the left-moving R-current anomaly through  $c_L = 6k_L$ .

Since  $c_L$  and  $c_R$  are determined in terms of the Chern-Simons coefficients they cannot receive any higher derivative corrections. From equation (6.25) this may seem surprising because we would expect the central charges to depend on all the derivative corrections.

In [127] the authors explain this puzzle. The first thing to note is that from the bulk point of view,  $(0, 4)$  supersymmetry prevents the addition of any higher derivative corrections except those that can be removed by a field redefinition. On the CFT side all the correlation functions of operators dual to the fields in the supergravity multiplet are determined completely in terms of  $c_L$  and  $c_R$ , the central charges of the left and right Virasoro algebras. Once we determine  $c_L$  and  $c_R$  via the gravitational and gauge Chern-Simons coefficients we can determine all correlation functions of the superconformal currents and therefore the boundary S-matrix of the supergravity fields. Now, two different theories with same boundary S-matrix must be related by a field redefinition. In other words two actions with  $(0, 4)$  supersymmetry and the same Chern-Simons terms must be related by a field redefinition.

## 6.2 Results from four dimensional black holes

We consider the D1-D5-KK-P black hole studied in section §2 (2). Since we are interested in answering some puzzles related to M theory on  $K3 \times T^2$ , raised in [128, 129, 130], we will proceed our analysis in the M-theory frame.

In type IIB compactified on  $K3 \times S^1 \times \tilde{S}^1$ , this black hole carries the following set of charges:  $Q_1$  D1-branes wrapping the circle  $S^1$ <sup>16</sup>,  $Q_5$  D5-branes wrapping  $K3 \times S^1$ ,  $\tilde{K}$  KK monopoles associated with the circle  $\tilde{S}^1$ ,  $n$  units of momentum along  $S^1$  and  $J$  units of momentum along  $\tilde{S}^1$ . First we use mirror symmetry on  $K3$  to map the D1-D5 to a D3-D3 system with  $Q_1$  D3-branes wrapping a 3-cycle  $\Sigma_2 \times S^1$  and  $Q_5$  D3-branes wrapping a 3-cycle  $\tilde{\Sigma}_2 \times S^1$ , with  $\Sigma_2$  and  $\tilde{\Sigma}_2$  a pair of dual 2-cycles of  $K3$ . We then make a T-duality along  $\tilde{S}^1$  to map the D3-branes to D4-branes and the KK monopoles to NS5-branes wrapped on  $K3 \times S^1$ . The momentum  $J$  is mapped to winding. We note the dual circle by  $\hat{S}^1$ . So far we have  $Q_1$  D4-branes on  $\Sigma_2 \times S^1 \times \hat{S}^1$ ,  $Q_5$  D4-branes on  $\tilde{\Sigma}_2 \times S^1 \times \hat{S}^1$ ,  $\tilde{K}$  NS5-branes on  $K3 \times S^1$ ,  $J$  fundamental strings wrapping  $\hat{S}^1$  and  $n$  units of momentum along  $S^1$  in the type IIA frame. We can now lift this configuration to M-theory on  $S_M^1$ , the M-theory circle. Altogether we have  $Q_1$  M5-branes wrapping the 5-cycle  $\Sigma_2 \times S^1 \times \hat{S}^1 \times S_M^1$ ,  $Q_5$  M5-branes wrapping  $\tilde{\Sigma}_2 \times S^1 \times \hat{S}^1 \times S_M^1$ ,  $\tilde{K}$  M5-branes wrapping  $K3 \times \hat{S}^1 \times S_M^1$ ,  $J$  M2-branes wrapping  $\hat{S}^1 \times S_M^1$  and  $n$  units of momentum along  $S^1$ .

The presence of spinning M2-branes in the background of M5-branes brings additional difficulties in the study of the low energy theory. Therefore we restrict to the case when M2-branes are absent by setting  $J = 0$ .

Our goal will be to compute the index associated with this black hole from a bulk perspective and then compare it with the index  $B_6$  that we determined in section §2, in the limit when we take  $n$  very large keeping the other charges finite. As explained before, in the limit of large radius  $S^1$ , the  $AdS_2 \times S^2$  near horizon of the four dimensional black hole in M-theory combines with the circle  $S^1$  to form a locally  $AdS_3 \times S^2$  factor [68, 35]. Moreover as we send the asymptotic radius  $S^1$  to infinity keeping the other moduli fixed the solution develops an intermediate  $AdS_3$  region where the black hole solution can be seen as a BTZ black hole. The entropy can be computed using the Cardy formula

$$S_{BH} \approx e^{2\pi\sqrt{nc_L/6}} \quad (6.37)$$

where  $c_L$  is the left central charge of the dual  $CFT_2$ . In the limit when all the charges are very large, we can use formula (6.25) to compute the leading contribution to the central charge  $c_L = 6Q_1Q_5\tilde{K}$  which comes, basically, from the Einstein-Hilbert term. This agrees with the Beckenstein-Hawking entropy  $S_{BH} \approx 2\pi\sqrt{Q_1Q_5\tilde{K}n}$ , [131, 132].

As explained in the previous section the central charge  $c_L$  can be computed exactly given the Chern-Simons coefficients in the bulk theory.

M-theory contains, already in eleven dimensions, higher derivative corrections. One of particular importance in this problem is a eighth derivative Chern-Simons term [133, 134]

$$\sim \int C^3 \wedge I_8(X) \quad (6.38)$$

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<sup>16</sup>The number  $Q_1$  is effectively the charge and not the number of D1-branes. Due to the presence of a Chern-Simons term of the form  $\int C^2 \wedge R \wedge R$  in the D5-brane world volume theory, there is an induced  $-N_5$  D1 charge, with  $N_5$  the number of D5 branes wrapping  $K3$ . Therefore the total D1 charge is  $Q_1 = N_1 - N_5$ , where  $N_1$  is the number of D1 branes.

where  $C^3$  is the three form gauge field of M theory. The eighth form is defined as

$$I_8(X) = \frac{1}{48} \left( p_2(X) - \frac{1}{4} p_1(X)^2 \right) \quad (6.39)$$

with  $X$  being the eleven dimensional space and  $p_n$  the  $n$ th Pontryagin class. Once we reduce the theory on  $K3 \times \tilde{S}^1 \times S_M^1$  a five dimensional gravitational Chern-Simons on  $AdS_3 \times S^2$  is generated [38, 35],

$$S_{CS} = \frac{1}{32\pi^2} \int_{AdS_3 \times S^2} \Omega_3(\Gamma) \wedge F \quad (6.40)$$

where  $F$  is the KK monopole gauge field strength and  $\Omega_3(\Gamma)$  is the gravitational Chern-Simons in  $AdS_3$ . Using the equation of motion  $\int_{S^2} F = 4\pi \tilde{K}$  we get

$$S_{CS} = \frac{1}{8\pi} \int_{AdS_3} \Omega_3(\Gamma). \quad (6.41)$$

From the coefficient of the gravitational Chern-Simons we can determine the difference between the central charges. That is,

$$c_L - c_R = 12\tilde{K}. \quad (6.42)$$

Concerning the gauge Chern-Simons it can have two possible origins. First, there is already in eleven dimensions a gauge Chern-Simons term of the form

$$\int C^3 \wedge F^4 \wedge F^4 \quad (6.43)$$

with  $C^3$  the M-theory three form and  $F^4$  its field strength. After reducing down to five dimensions it originates a five dimensional Chern-Simons on  $AdS_3 \times S^2$  of the form

$$\int C^1 \wedge F^2 \wedge F^2 \quad (6.44)$$

with  $F^2 = dC^1$  and  $C^1$  is the reduction of  $C^3$  on  $\tilde{S}^1 \times S_M^1$ . The reduction of a term of this type down to  $AdS_3$  was analyzed in [135]. The authors do a careful treatment of the  $SU(2)$  gauge fields which result from gauging the  $S^2$  isometries. Roughly, they consider a solution where  $S^2$  is fibered over  $AdS_3$  with the fibers being the  $SU(2)$  connections,

$$ds^2 = ds_{AdS_3}^2 + \sum_{i=1}^3 (dy^i - A_j^i(x) dx^j)^2, \quad \sum_{i=1}^3 (dy^i)^2 = 1. \quad (6.45)$$

where  $A_j^i$  is the  $SO(3)$  R-connection. At infinity, that is, near the boundary  $A_j^i$  vanishes and we recover  $AdS_3 \times S^2$ . A careful treatment of the fluxes over a gauged  $S^2$  has to be considered in this case. They found that the gauge field  $C^1$  that carries magnetic charge on  $S^2$  will not be invariant under an  $SU(2)$  gauge transformation of the  $S^2$  fibers, inducing a  $SU(2)$  gauge Chern-Simons via the term (6.44). This term will be responsible for an anomalous conservation of the  $SU(2)$  right-moving R-current on the boundary theory. Moreover, using  $(0, 4)$  supersymmetry

we determine  $c_R = 6k_r$  where  $k_R$  is the R-current anomaly. Since the Chern-Simons term in (6.44) is a two derivative term, the contribution to the central charge will correspond to the leading supergravity approximation,

$$c_R = 6Q_1Q_5K + \dots \quad (6.46)$$

The other possible contribution comes from the eighth derivative Chern-Simons term in eleven dimensions (6.38). Because this term is higher order in derivatives it will give subleading corrections to  $c_R$ . The gauging of  $S^2$  induces an additional contribution in the tangent space connection  $\Gamma$  of  $AdS_3 \times S^2$ . The total connection is the direct sum of the  $SO(2,1)$  connection  $\Gamma_{AdS_3}$  of  $AdS_3$  and the  $SO(3)$  connection  $A$  associated with the sphere:  $\Gamma = \Gamma_{AdS_3} \oplus A$ . The five dimensional gravitational Chern-Simons term decomposes as [35, 136, 137]

$$\Omega_3(\Gamma_{AdS_3 \times S^2}) = \Omega_3(\Gamma_{AdS_3}) + \Omega_3(A). \quad (6.47)$$

Therefore reducing (6.38) on  $K3 \times \hat{S}^1 \times S_M^1 \times S^2$  originates an additional Chern-Simons term

$$\frac{\tilde{K}}{2\pi} \int_{AdS_3} \Omega_3(A_R) \quad (6.48)$$

where the gauge field  $A_R$  is in the adjoint of  $SU(2)^{17}$ . This term generates an additional contribution of  $12\tilde{K}$  to the central charge  $c_R$ . Together with the leading contribution (6.46) we get

$$c_R = 6Q_1Q_5\tilde{K} + 12\tilde{K}. \quad (6.49)$$

Now using the fact that  $c_L - c_R = 12\tilde{K}$  we conclude that the central charge  $c_L$  is given by

$$c_L = 6Q_1Q_5\tilde{K} + 24\tilde{K}. \quad (6.50)$$

If we consider the most general case of an M5-brane wrapping a five-cycle  $P \times S^1$ , where  $P$  is a divisor of  $M = K3 \times S_M^1 \times \tilde{S}^1$ , we find [35, 38, 128]

$$c_R = \int_M \tilde{P} \wedge \tilde{P} \wedge \tilde{P} + \frac{1}{2} \tilde{P} \wedge c_2(M), \quad c_L = \int_M \tilde{P} \wedge \tilde{P} \wedge \tilde{P} + \tilde{P} \wedge c_2(M) \quad (6.51)$$

where  $\tilde{P}$  is the 2-cycle dual to  $P$  in  $M$  and  $c_2(M)$  is the second chern class of  $M$ . In terms of a basis of four cycles  $\sigma^a$  of  $M$  and the charge vector  $q_a$ , the divisor  $P$  is represented as  $P = q_a \sigma^a$ . In this case we have  $P = K\sigma(K3) + Q_1\sigma(\Sigma^2 \times S^1 \times \tilde{S}^1) + Q_5\sigma(\tilde{\Sigma}^2 \times S^1 \times \tilde{S}^1)$ .

We could wonder if there are any other terms in the eleven dimensional theory that could possibly contribute with additional gravitational or gauge Chern-Simons terms.

To study this possibility we use the scaling argument developed in [138]. This is a useful way to track at which order in perturbation theory potential higher derivative terms can arise.

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<sup>17</sup>In rewriting the  $SO(3)$  connection  $A = \frac{1}{2}A^{ij}J^{ij}$ , where  $J^{ij}$  are the  $SO(3)$  generators, in terms of  $SU(2)$  generators  $J^a$  as  $A = A^a J^a$  there is an additional factor of four in the Chern-Simons term due to the fact that  $\text{Tr}(J^{ij})^2 = 4\text{Tr}(J^a)^2$

Say we take an extremal black hole carrying NS-NS electric  $q_{NS-NS}^{ele}$  and magnetic  $q_{NS-NS}^{mag}$  charges, and RR charges  $q_{RR}$ . The tree level IIA/B string theory action has a scaling symmetry under which the dilaton  $\phi$  gets shifted by a constant  $-\ln \lambda$ , the NS-NS fields remain invariant and the RR fields are multiplied by  $\lambda$ . The effect of this scaling is to multiply the action by  $\lambda^2$ . In terms of the charges, this corresponds to multiply by  $\lambda^2$  the NS-NS electric charges and by  $\lambda$  the RR charges while leaving invariant the magnetic charges. The formula (6.25) relates the central charge to the string theory action computed in the near horizon background. Hence, the central charge as a function of the charges will have the same scaling symmetry as the tree level action, that is,

$$c^0(\lambda^2 q_{NS-NS}^{ele}, q_{NS-NS}^{mag}, \lambda q_{RR}^{ele}) = \lambda^2 c^0(q_{NS-NS}^{ele}, q_{NS-NS}^{mag}, q_{RR}^{ele}) \quad (6.52)$$

where  $c^0$  denotes the tree level contribution to the central charge. In the same spirit we can keep track of additional l-loop contributions through

$$c^l(\lambda^2 q_{NS-NS}^{ele}, q_{NS-NS}^{mag}, \lambda q_{RR}^{ele}) = \lambda^{2-2l} c^l(q_{NS-NS}^{ele}, q_{NS-NS}^{mag}, q_{RR}^{ele}) \quad (6.53)$$

where  $c^l$  is the l-loop contribution to the central charge. In the case of the D1-D5-KK system we have  $Q_1$  and  $Q_5$  RR charges and  $\tilde{K}$  NS-NS magnetic charge. Since the central charge is an integer we can write it as a polynomial in the charges. Terms linear in  $Q_1$  or  $Q_5$  are not allowed because they would correspond to a 1/2-loop contribution which is absent in closed string theory. Moreover the dependence on the RR charges must come in the form  $Q_1 Q_5$  if we want to respect duality. The contribution  $6Q_1 Q_5 \tilde{K}$  to the central charge scales as  $\lambda^2$  since it comes from the tree level action. The remaining  $12\tilde{K}$  corresponds to one-loop contribution because it doesn't scale. Indeed the term (6.38) is generated at one-loop in string theory [134, 133]. Though all this analysis, we can still ask whether additional one-loop contributions to the gravitational or gauge Chern-Simons can arise after compactification on  $M \times S^2$ . A priori we can not rule out such possibility. Equations (6.26) and (6.27) show that additional one-loop Chern-Simons terms in  $AdS_3$  can be generated. For example in  $AdS_5$  there is a one-loop generated  $SU(4)$  gauge Chern-Simons term that is responsible for a constant  $-1$  correction to the leading  $N^2$   $SU(4)$  R-symmetry anomaly in  $SU(N)$  SYM [126]. If we denote by  $a_L$  and  $a_R$  possible one-loop contributions to  $c_L$  and  $c_R$  respectively, we have

$$c_L = 6Q_1 Q_5 \tilde{K} + 24\tilde{K} + a_L, \quad c_R = 6Q_1 Q_5 \tilde{K} + 12\tilde{K} + a_R \quad (6.54)$$

To determine the full contribution to the black hole index we have to combine the bulk results with the exterior modes contribution. The total central charge which controls the asymptotic growth of the index is

$$c_L^{macro} = 6Q_1 Q_5 \tilde{K} + 24\tilde{K} + a_L + c_L^{eff} \quad (6.55)$$

In the next section we will see that the effect of  $c_L^{eff}$  is to cancel the constant one-loop contribution  $a_L$ .

### 6.3 Exterior modes contribution

The central charge of the left Virasoro algebra can be determined in terms of the coefficients of the gravitational and  $SU(2)$  gauge Chern-Simons present in the bulk action of the intermediate  $AdS_3$  geometry. The left central charge is given by a simple formula

$$c_L = c_{bulk}^{grav} + 6k_R \quad (6.56)$$

where  $c_{bulk}^{grav} = c_L - c_R$  is proportional to the coefficient of the gravitational Chern-Simons and  $k_R$  is the R-current anomaly which is proportional to the coefficient of the  $SU(2)$  gauge Chern-Simons term. Part of the answer came from integrating Chern-Simons terms already existing in eleven dimensional M-theory on  $K3 \times \tilde{T}^2 \times S^2$ , down to five dimensions. We also argued that additional one-loop constant contributions generated after compactification were possible.

Imagine that instead of doing a reduction on  $K3 \times \tilde{T}^2 \times S^2$  we do this in the asymptotic region where the eleven dimensional geometry looks like  $K3 \times \tilde{T}^2 \times \mathbb{R}^5$ . In a vast region of space-time, namely for  $1 \ll r_1 \ll r \ll r_2$ , the space looks like  $K3 \times \tilde{T}^2 \times S^2 \times \mathbb{R}^3$ , where  $\mathbb{R}^3$  contains the time coordinate, the radius  $r$  and the coordinate  $y$  corresponding to the circle  $S^1$ . We can now compactify the eleven dimensional theory on  $K3 \times \tilde{T}^2 \times S^2$  and compute the coefficients of the gravitational and gauge Chern-Simons with support on  $\mathbb{R}^3$ . The calculation will be identical to what we have done in the previous section except that in this case the one-loop corrections are not generated after compactification.

These coefficients will be identical to (6.54) except for the constant shifts. On  $\mathbb{R}^3$  we denote by  $c_{grav}^{asympt}$  and  $k_R^{asympt}$  the gravitational and gauge Chern-Simons coefficients. For the D1-D5-KK system we have

$$c_{grav}^{asympt} = 12\tilde{K}, \quad k_R^{asympt} = Q_1 Q_5 \tilde{K} + 2\tilde{K} \quad (6.57)$$

The difference between the anomaly coefficients computed at asymptotic infinity and those computed at the intermediate  $AdS_3$  must be accounted by the exterior modes which live in between those two regions. That is,

$$c_{grav}^{asympt} = c_{grav}^{bulk} + c_{grav}^{exterior}, \quad k_R^{asympt} = k_R^{bulk} + k_R^{exterior} \quad (6.58)$$

where  $c_{grav}^{exterior}$  and  $k_R^{exterior}$  denote the contributions to the gravitational and  $SU(2)$  current anomalies from the exterior modes. Since both  $c_{grav}^{asympt}$  and  $k_R^{asympt}$  are free from constant shifts, the effect of  $c_{grav}^{exterior}$  and  $k_R^{exterior}$  is to cancel the one-loop contributions  $a_L$  and  $a_R$  in (6.54), that is,

$$6k_R^{exterior} + a_R = 0, \quad c_{grav}^{exterior} + a_L = 0 \quad (6.59)$$

Ultimately we are interested in the total index which is controlled by an effective central charge which we denote by  $c_L^{macro}$  (6.55). This effective central charge is the sum of the bulk central charge  $c_L$  and an effective central charge  $c_L^{eff}$  that controls the growth of the index for the exterior modes. For the bulk degrees of freedom we used holography to determine the left central charge in terms of the Chern-Simons, that is,  $c_L = c_{grav}^{bulk} + 6k_R^{bulk}$ , while for the exterior modes, a priori there's no relation between  $c_L^{eff}$ ,  $c_{grav}^{exterior}$  and  $k_R^{exterior}$  because we don't have



a dual description of the theory. Once we find the relation between these coefficients we can relate  $c_L^{macro}$  to the known quantities  $c_{grav}^{asympt}$  and  $k_R^{asympt}$ .

In the next we show, based on certain assumptions, that the following relation holds

$$c_L^{eff} = c_{grav}^{exterior} + 6k_R^{exterior}. \quad (6.60)$$

Surprisingly it is identical to the relation (6.56) valid for the bulk theory. In general these exterior modes are associated with the center of mass degrees of freedom of a brane system like the singleton in  $AdS_5$  which describes the  $U(1)$  factor of the  $U(N)$  SYM. For the exterior modes we are not able to identify the R-symmetry of the two dimensional theory with the rotation group of  $S^2$  since the scalars that describe transverse motion are not chiral under rotations. For example in the case of the D1-D5 system they describe motion in the transverse  $\mathbb{R}^4$ . The  $SO(4)$  R-symmetry of the dual  $CFT_2$  then acts non-chirally on the bosons as  $SO(4)$  rotations. Since the R-symmetry is a chiral symmetry,  $SO(4)$  cannot be the R-symmetry of the superconformal  $\mathbb{R}^4$  sigma model [52]. If we were able to identify the R-symmetry of the exterior superconformal theory with the rotation group of the transverse space then  $k_R^{exterior}$  would correspond to the R-symmetry current anomaly  $k_R$ . In addition we could use  $(0,4)$  supersymmetry to compute the right super-Virasoro central charge via  $c_R = 6k_R$  and then conclude  $c_L^{exterior} = c_{grav}^{exterior} + 6k_R^{exterior}$ .

In attempting to derive the formula (6.60) we make the assumptions

1. The exterior modes consist of free massless scalars and fermions belonging to singlet and/or spinor representations of  $SU(2)_L \times SU(2)_R$ .
2. The scalar modes that transform in the vector representation of  $SO(4)$  are non-chiral. Under this assumption the contribution of the scalars to the  $SU(2)_L$  or  $SU(2)_R$  current anomalies always vanish.

This is basically the content of the  $\mathbb{R}^4$  sigma model we described in section §2 (2) when studying the microscopic degrees of freedom of the D1-D5-KK associated with the center of mass motion of the D1-D5 system.

Note that we are considering the most general case by allowing the fields to be charged under an additional  $SU(2)_L$ . This would be relevant for the case when the black hole has a  $S^3$  factor in the near horizon geometry, in which case the R-symmetry is  $SU(2)_L \times SU(2)_R$ .

In addition the  $1+1$  dimensional CFT describing the exterior mode part has  $(0,4)$  supersymmetry. This follows from the supersymmetry of the solution outside the  $AdS_3$  region.

Both the anomaly coefficients  $k_R^{exterior}$  and  $c_{grav}^{exterior} = c_L^{exterior} - c_R^{exterior}$  can be extracted by reading the quantum numbers of the fields. For example a complex right-moving fermion charged under  $SU(2)_R$  will contribute with  $1/2$  to the right-moving current anomaly  $k_R$  while a left-moving fermion charged under the same group contributes with  $-1/2$  and vice-versa for  $SU(2)_L$ . As usual, the central charge is given by the number of bosons plus half the number of fermions. To compute  $c_L^{eff}$  we have to read the asymptotic behaviour of the index  $B_{6^{exterior}} = -1/6! \text{Tr}(-1)^{2J_R} (2J_R)^6$  in the limit of large  $n$ . We use instead  $B_{6^{exterior}} = \text{Tr}(-1)^{2J_R}$

where integration over fermion zero modes has been carried out. Since the theory has  $(0, 4)$  supersymmetry, BPS condition forces the right-moving excitations to be in the ground state.

Consider the example of left-moving  $N_b$  scalars and  $N_f$  fermions uncharged under  $SU(2)_R$ . The partition function can be easily computed to give

$$\text{Tr}(-1)^{2J_R} q^{L_0} = \sum d(n) q^n = \frac{1}{q^{(N_b - N_f)/24}} \frac{\prod_m (1 + q^m)^{N_f}}{\prod_m (1 - q^m)^{N_b}}, \quad q = e^{2\pi i \tau}.$$

For large  $n$  the index grows as  $d(n) \approx e^{2\pi \sqrt{nc_L/6}}$  with  $c_L = N_b + 1/2 N_f$ . In this case  $c_L^{eff} = c_L$ . Now consider the same system but with the fermions charged under  $SU(2)_R$ . The partition function is now

$$\sum d(n) q^n = \frac{1}{q^{(N_b - N_f)/24} \prod_m (1 - q^m)^{N_b - N_f}}, \quad q = e^{2\pi i \tau}.$$

In this case  $d(n) \approx e^{2\pi \sqrt{nc_L^{eff}/6}}$ , for large  $n$ , with  $c_L^{eff} = N_b - N_f \neq c_L$ . The left-moving fermions all together contribute with  $-N_f/4$  to the  $SU(2)_R$  current anomaly. On the supersymmetric side we have  $N$  bosons and  $N$  fermions both charged under  $SU(2)_R$ . Their contribution is  $c_R = 3N/2$  for the central charge and  $N/4$  for the right R-current anomaly. The total anomaly becomes  $k_R = N/4 - N_f/4$ . Note that in this case  $c_R \neq 6k_R$  consistent with our assumptions. A straightforward calculation shows that  $c_L^{eff} = c_{grav} + 6k_R$ . Note that in this example  $c_L^{eff}$  can be negative provided  $N_f > N_b$  which is possible because the left-moving sector is not supersymmetric.

We could repeat this analysis for many other examples though we would arrive at the same conclusion

$$c_L^{eff} = c_{grav} + 6k_R. \quad (6.61)$$

Unfortunately we lack a physical understanding of this result.

We are now in a position to compute the total index in terms of the anomaly coefficients measured at asymptotic infinity.

Using equations (6.61) and (6.58) we compute the total index

$$c_L^{macro} = c_L^{bulk} + c_L^{eff} = c_{grav}^{bulk} + c_{grav}^{exterior} + 6k_R^{bulk} + 6k_R^{exterior} = c_{grav}^{asympt} + 6k_R^{asympt} \quad (6.62)$$

from which we conclude that *the coefficient  $c_L^{macro}$  that controls the growth of the total index is given in terms of the coefficients of the gravitational and gauge Chern-Simons computed at asymptotic infinity.* For the case of the D1-D5-KK this translates to

$$c_L^{macro} = 6Q_1 Q_5 \tilde{K} + 24\tilde{K} \quad (6.63)$$

We end this section by making some remarks on equation (6.58) based on anomaly inflow [136, 137].

It was pointed out some time ago [139] that the anomalous conservation of the charge current in a string, due to the presence of chiral fermion zero modes, should be cancelled by an inflow of charge current from the exterior. This mechanism is called anomaly inflow. It was

extremely useful in explaining both the tangent and normal bundle anomalies of an M5-brane [137, 136]. In the even dimensional world volume theory of the M5-brane there are chiral fields which render both the  $SO(5, 1)$  and  $SO(5)$ , respectively, tangent and normal diffeomorphisms, anomalous. The associated charges will therefore be anomalously conserved. This means that the total charge in the world volume theory will vary with time unless there is an inflow of a charge current from the exterior bulk. This was consistent with the fact that there is already in the eleven dimensional supergravity a eighth derivative Chern-Simons coupling of the form

$$\sim \int_{11} C_3 \wedge I_8(X)$$

where  $C_3$  is the three form gauge field of M-theory and  $I_8(X)$  is a eight form constructed out of the Riemann tensor. We had made use of this term to derive the Chern-Simons terms in the five dimensional theory of black holes in M-theory (6.38). In this context the Chern-Simons terms are computed at asymptotic infinity and therefore the anomaly coefficients are free from constant one-loop corrections. Equations (6.58) are equivalent to anomaly inflow.

## 6.4 Microscopic results

We borrow the microscopic results from section §2 (2).

For a primitive dyon, that is, a dyon for which  $\gcd(Q \wedge P) = 1$  we saw that the index was given by the fourier coefficient of the inverse of the Siegel modular form  $\Phi_{10}$ , that is,

$$B_6(Q^2, P^2, Q.P) = (-1)^{Q.P+1} \int_{\mathcal{C}} d\rho d\sigma dv \frac{e^{-i\pi\rho Q^2 - i\pi\sigma P^2 - 2\pi i v Q.P}}{\Phi_{10}(\rho, \sigma, v)} \quad (6.64)$$

While in the more general case with  $I \geq 1$ , for dyons  $(Q, P) = (IQ_0, P_0)$  with  $\gcd(Q_0 \wedge P_0) = 1$ , the index was given by

$$B_6(Q, P) = \sum_{s|I} s d_1 \left( \frac{Q^2}{s^2}, P^2, \frac{Q.P}{s} \right), \quad (6.65)$$

where  $d_1 \left( \frac{Q^2}{s^2}, P^2, \frac{Q.P}{s} \right)$  is computed from the primitive answer.

We are interested in the behaviour of  $B_6$  in the limit of large  $Q^2 = 2nK$ , finite  $P^2 = 2Q_1Q_5$  and  $Q.P = 0$ . We use an asymptotic expansion of the primitive answer derived in [26] which has the form (2.39)

$$B_6 \simeq \int \frac{d^2\tau}{\tau_2^2} e^{-F(\tau_1, \tau_2)} \quad (6.66)$$

with

$$\begin{aligned} F(\tau_1, \tau_2) = & -\ln \left( 26 + \frac{\pi}{\tau_2} (Q^2 + P^2 |\tau|^2) \right) - \frac{\pi}{2\tau_2} (Q^2 + P^2 |\tau|^2) \\ & + 24 \ln \eta(\tau) + 24 \ln \eta(-\bar{\tau}) + 12 \ln(\tau_2). \end{aligned} \quad (6.67)$$

We compute this integral using a saddle point approximation. Due to the symmetry  $\tau_1 \rightarrow -\tau_1$  of the free energy  $F$  we can set  $\tau_1 = 0$  at the saddle point. At the same time we use the ansatz

that  $\tau_2$  becomes very large at the saddle point to simplify  $F(\tau_1, \tau_2)$

$$F(0, \tau_2) \simeq -\frac{\pi}{2\tau_2}(Q^2 + P^2\tau_2^2) - 4\pi\tau_2. \quad (6.68)$$

From this expression we determine the value of  $\tau_2$  at the saddle point

$$\tau_2^* = \sqrt{\frac{Q^2}{P^2 + 8}} \quad (6.69)$$

In the limit considered  $\tau_2^*$  becomes very large which justifies our assumption. We can now estimate the asymptotic growth of the index

$$\ln B_6 \simeq \pi\sqrt{Q^2(8 + P^2)} \quad (6.70)$$

For the charge configuration of our problem we have  $Q^2 = 2n\tilde{K}$  and  $P^2 = 2Q_1Q_5$  which gives

$$\ln B_6 \simeq 2\pi\sqrt{nK(4 + Q_1Q_5)} \quad (6.71)$$

This is in perfect agreement with the macroscopic derivation (6.63).

#### 6.4.1 MSW string

Here we are interested on the derivation of the microscopic index using directly the data from the 1+1 low energy theory of the M5-brane on the divisor  $P$ .

As explained before the D1-D5-KK system can be mapped to a M5-brane wrapping a five cycle  $P \times S^1$  with  $P$  a divisor in  $M = K3 \times \hat{S}^1 \times S^1_M$ . If the size of the circle  $S^1$  is much larger than the typical size of  $M$ , then theory which describes the low energy fluctuations of a M5-brane on  $P \times S^1$  is a 1 + 1 dimensional (0, 4) SCFT.

The BPS states in this theory involve the left-moving excitations. The growth of the degeneracy of these states is given by a Cardy formula determined in terms of the central charge  $c_L^{micro}$  of the left Virasoro algebra. For a theory with  $N_b$  left-moving bosons and  $N_f$  left-moving fermions the central charge is  $c_L^{micro} = N_b + N_f/2$ . Since our interest is the index  $B_6$  instead of the degeneracy, the computation goes differently. After integrating over the fermionic zero modes,  $B_6$  reduces to the Witten index  $\text{Tr}(-1)^F$ . While  $(-1)^F$  does not affect the contribution from a bosonic oscillator it can change that of a fermion. The growth of the index is now controlled by an effective central charge [140] given by

$$c_{L,eff}^{micro} = N_b - N_f. \quad (6.72)$$

Note that if  $N_f = 0$  then  $c_{L,eff}^{micro} = c_L^{micro}$ .

According to the analysis of [128], the number of bosons in the low energy theory is

$$\begin{aligned} N_b^L &= d_p(P) + b_2^- + 3, \\ N_b^R &= d_p(P) + b_2^+ + 3 \end{aligned} \quad (6.73)$$

The upper indices  $L, R$  denote left and right sectors,  $d_p(P)$  is the dimension of the moduli space of deformations of the divisor  $P$  inside  $M$ , 3 accounts for the center of mass translations and  $b_2^-, b_2^+$  denote the number of self and anti-self dual two forms of  $P$ . These scalars originate from the reduction of the two form living on the world-volume of the M5-brane. For the fermions we have

$$\begin{aligned} N_f^L &= 4h_{1,0}(P), \\ N_f^R &= 4h_{2,0}(P) + 4 \end{aligned} \quad (6.74)$$

For an ample divisor  $P$  the authors in [128] gave an expression for  $d_p(P)$  when  $M$  is a manifold without one-cycles. In this case  $M = K3 \times S_M^1 \times \tilde{S}^1$  contains two one-cycles. Therefore we proceed differently following [129]. On a Kahler manifold we have the following relations

$$b_2 = b_2^+ + b_2^- = 2h_{2,0} + h_{1,1}, \quad b_2^- = h_{1,1} - 1 \quad (6.75)$$

Using these results in equations (6.73) and (6.74) together with the fact that for the right-moving sector supersymmetry implies  $N_b^R = N_f^R$ , we find

$$d_p(P) = 2h_{2,0}. \quad (6.76)$$

For the left-moving fields we get

$$N_b^L = 2h_{2,0}(P) + h_{1,1} + 2 = b_{\text{even}}(P), \quad N_f^L = 4h_{1,0} = b_{\text{odd}}(P). \quad (6.77)$$

The effective central charge  $c_{L,\text{eff}}^{\text{micro}} = N_b^L - N_f^L$  is just the Euler character of  $P$ , that is,  $c_{L,\text{eff}}^{\text{micro}} = b_{\text{even}} - b_{\text{odd}} = \chi(P)$ .

This has a simple expression in terms of the two-cycle  $\tilde{P}$  dual to  $P$  in  $M$

$$c_{L,\text{eff}}^{\text{micro}} = \chi(P) = \int_M \tilde{P} \wedge \tilde{P} \wedge \tilde{P} + \tilde{P} \wedge c_2(M) \quad (6.78)$$

where  $c_2(M)$  is the second chern class of  $M$ . In the particular example of the D1-D5-KK the divisor is  $P = \tilde{K}\sigma(K3) + Q_1\sigma(\Sigma^2 \times S^1 \times \tilde{S}^1) + Q_5\sigma(\tilde{\Sigma}^2 \times S^1 \times \tilde{S}^1)$  where  $\sigma$  denotes a four cycle in  $M$ . Substituting in the formula above we get

$$c_{L,\text{eff}}^{\text{micro}} = 6Q_1Q_5\tilde{K} + 24\tilde{K}. \quad (6.79)$$

in perfect agreement with both the microscopic result computed from  $\Phi_{10}^{-1}$  and the macroscopic one computed from the coefficients of the Chern-Simons terms. If we computed the physical left central charge that controls the degeneracy instead of the index we would find

$$c_L^{\text{micro}} = N_b^L + N_f^L/2 = 6Q_1Q_5\tilde{K} + 24\tilde{K} + 6 \quad (6.80)$$

where we used  $N_f^L = 4h_{1,0}(P) = 4h_{1,0}(M) = 4$ . As observed in [129, 130], (6.80) fails to agree with the macroscopic result (6.63). Hence we see that the apparent puzzle arose from comparing the microscopic degeneracy with the microscopic index.

## 6.5 Five dimensional black holes

The analysis goes more or less in the same way as in the four dimensional case. The main difference resides on the fact that the five dimensional black hole can carry angular momentum without breaking supersymmetry.

The spatial rotation group in five dimensions is  $SU(2)_L \times SU(2)_R$ . We denote by  $J_L$  and  $J_R$  the  $U(1)$  generators of both factors. Among all the supersymmetry generators of the theory half belong to the  $(1_L, 2_R)$  and the other half to  $(2_L, 1_R)$  representations of  $SU(2)_L \times SU(2)_R$ . We choose the convention that for a state preserving four supersymmetries the unbroken generators are in the  $(1_L, 2_R)$  representation. Moreover  $k$  broken supersymmetries give rise to  $k$  fermion zero modes. For those which are charged under  $SU(2)_R$ , we insert a power of  $(2J_R)$  for each pair of fermion zero modes to render the index non vanishing.

For a five dimensional black hole in  $\mathcal{N} = 4$  string theory, like in the case of the D1-D5-P in type IIB on  $K3 \times S^1$  [53], the solution carries two complex fermion zero modes in the  $(1_L, 2_R)$  representation. If we consider the same brane system but in type IIB on  $(T^4 \times S^1)$  [27], we have six complex fermion zero modes instead of two.

To capture the BPS states we use the spacetime index

$$B_2 = -\frac{1}{2} \text{Tr}(-1)^{2J_R - 2J_L} (2J_R)^2 \quad (6.81)$$

where we sum over  $J_R$  and fix the angular momentum  $J_L$  and the remaining charges. For simplicity we shall use the index defined as

$$C_2 = -\frac{1}{2} \text{Tr}(-1)^{2J_R} (2J_R)^2 \quad (6.82)$$

which is related to  $B_2$  by a simple operation  $B_2 = (-1)^{2J_L} C_2$ .

The near horizon geometry of these black holes has locally an  $AdS_3 \times S^3$  factor. We proceed similarly as in section §6.2. By sending the attractor radius of the circle  $S^1$  to infinity keeping the other charges finite, we combine the circle  $S^1$  with  $AdS_2$  to form a locally  $AdS_3$ . At the same we take the asymptotic radius to infinity and keep the other moduli fixed to get an intermediate  $AdS_3$  region where the solution can be embedded as an extremal BTZ black hole. Since we keep fixed the angular momentum  $J$  while taking the asymptotic radius  $R$  of the circle to infinity the physical angular momentum  $J/R$  vanishes restoring the  $S^3$  symmetry as seen from an asymptotic observer.

As before we consider the contribution to the index from the bulk and exterior degrees of freedom. The index  $C_2$  becomes

$$C_2 = -\frac{1}{2} \text{Tr}(-1)^{2J_R^{bulk} + 2J_R^{exterior}} (2J_R^{bulk} + 2J_R^{exterior})^2 \quad (6.83)$$

$$= -\frac{1}{2} \text{Tr}(-1)^{2J_R^{bulk}} \text{Tr}(-1)^{2J_R^{exterior}} (2J_R^{exterior})^2 \quad (6.84)$$

$$= \sum_{\substack{q_1 + q_2 = q \\ J_1 + J_2 = J_L}} C_{bulk}(q_1, J_1) C_{2exterior}(q_2, J_2) \quad (6.85)$$

where we have defined  $C_{bulk} = \text{Tr}(-1)^{2J_R^{bulk}}$  and  $C_{2^{exterior}} = -\frac{1}{2}\text{Tr}(-1)^{2J_R^{exterior}}(2J_R^{exterior})^2$ . We proceed by constructing the partition function

$$\tilde{C}_2(\tau, z) = \sum_{n, J_L} C_2(n, J_L) e^{2\pi i \tau + 2\pi i J_L z} \quad (6.86)$$

$$= \tilde{C}_{bulk}(\tau, z) \tilde{C}_{2^{exterior}}(\tau, z) \quad (6.87)$$

In the  $CFT_2$  dual to the bulk  $AdS_3$  we can identify the  $SU(2)_L$  and  $SU(2)_R$  rotation symmetries with the left and right R-symmetries. In the same spirit of (6.11),  $\tilde{C}_{bulk}(\tau)$  as the small  $\tau$  behavior

$$\tilde{C}_{bulk}(\tau) \approx e^{\frac{\pi i c_L}{12\tau} - 2\pi i \frac{k_L z^2}{\tau}} \quad (6.88)$$

where  $c_L$  is the central charge of the left Virasoro algebra and  $k_L$  is the  $SU(2)_L$  R-current anomaly.

Using a saddle point approximation, the entropy of the rotating black hole is given by

$$\ln C_{bulk} \approx 2\pi \sqrt{c_L/6 \left( n - \frac{J_1^2}{4k_L} \right)} \quad (6.89)$$

in agreement with supergravity computations [53]. We shall also argue that for the exterior modes the corresponding partition function has a similar behaviour for small  $\tau$ , that is,

$$\tilde{C}_{2^{exterior}}(\tau) \approx e^{\frac{\pi i c_L^{eff}}{12\tau} - 2\pi i \frac{k_L^{eff} z^2}{\tau}} \quad (6.90)$$

where  $c_L^{eff}$  and  $k_L^{eff}$  shouldn't be confused neither with the left central charge nor with the current anomaly. These are coefficients that control the asymptotic growth and should be computed case by case. Their physical origin is not known for the moment. As in section §6.1 these coefficients must be determined directly from the index. Joining both bulk (6.88) and exterior (6.90) contributions, the partition function  $\tilde{C}_2(\tau, z)$  has small  $\tau$  behaviour

$$\tilde{C}_2(|\tau| \ll 1) \approx e^{\frac{\pi i (c_L + c_L^{eff})}{12\tau} - 2\pi i \frac{(k_L + k_L^{eff}) z^2}{\tau}}. \quad (6.91)$$

Performing a saddle point approximation, the growth of the total index is given by

$$\ln C_2 \approx 2\pi \sqrt{c_L^{macro}/6 \left( n - \frac{J_1^2}{4k_L^{macro}} \right)} \quad (6.92)$$

where we have defined  $c_L^{macro} = c_L + c_L^{eff}$  and  $k_L^{macro} = k_L + k_L^{eff}$ .

So far we have analysed the dependence on the index in terms of microscopic quantities like the central charges or the R-current anomalies. Since we don't know the details of the dual  $CFT_2$  we use the same technology of section §6.2 to determine  $c_L$  and  $k_L$  using the coefficients of the Chern-Simons terms in the bulk of  $AdS_3$ . In this case we need to consider in addition the  $SU(2)_L$  gauge Chern-Simons term to determine  $k_L$ .

For the exterior modes we have to compute  $c_L^{eff}$  and  $k_L^{eff}$  directly from the index. The analysis goes in the same manner as in the four dimensional case. We study case by case two dimensional field theories which obeying the assumptions (6.3) may correspond to an exterior contribution. We found the following relations

$$c_L^{eff} = c_{grav}^{exterior} + 6k_R^{exterior} \quad (6.93)$$

$$k_L^{eff} = k_L^{exterior} \quad (6.94)$$

The coefficient  $c_L^{eff}$  is the sum of  $c_{grav}^{exterior} = c_L^{exterior} - c_R^{exterior}$  and the  $SU(2)_R$  current anomaly  $k_R^{exterior}$  while  $k_L^{eff}$  equals the  $SU(2)_L$  current anomaly.

The anomalous contributions from the bulk and the exterior modes must be combined to give the anomaly coefficients measured at asymptotic infinity

$$c_{grav}^{asympt} = c_{grav}^{bulk} + c_{grav}^{exterior} \quad (6.95)$$

$$k_R^{asympt} = k_R^{bulk} + k_R^{exterior} \quad (6.96)$$

$$k_L^{asympt} = k_L^{bulk} + k_L^{exterior} \quad (6.97)$$

These together with the previous results give

$$c_L^{macro} = c_L + c_L^{eff} = c_{grav}^{asympt} + 6k_R^{asympt} \quad (6.98)$$

$$k_L^{macro} = k_L + k_L^{eff} = k_L^{asympt} \quad (6.99)$$

Since both  $c_{grav}^{asympt}$  and  $k_R^{asympt}$  don't receive one-loop constant corrections we arrive at the same conclusion that the effect of the exterior mode contribution is to cancel the one-loop contributions in the bulk anomaly coefficients which appear after compactification. The coefficients  $c_L^{macro}$  and  $k_L^{macro}$  are the same as  $c_L$  and  $k_L$  except for the constants shifts.

The asymptotic growth of the total index, as in the case of four dimensional black holes, *is determined in terms of the coefficients of the gravitational and gauge Chern-Simons computed at asymptotic infinity*. We find perfect agreement with the microscopic answer for the BMPV black hole [35] computed from the low energy dynamics of the D1-D5 system on  $K3$  [27, 26, 30].

### 6.5.1 Microscopic derivation

For the D1-D5 black hole in type IIB on  $K3 \times S^1$  with  $Q_1$  D1-branes wrapping  $S^1$ ,  $Q_5$  D5-branes wrapping  $K3 \times S^1$  and  $n$  units of momentum on  $S^1$ , the microscopic index is given by the formula [27, 35, 69]

$$C_2(n, Q_1 Q_5, J) = (-1)^{J+1} \int_0^1 d\rho \int_0^1 d\sigma \int_0^1 dv (e^{\pi i v} - e^{-\pi i v})^4 \frac{\eta(\rho)^{24}}{\Phi_{10}(\rho, \sigma, v)} e^{-2\pi i (n\rho + \sigma Q_1 Q_5 + Jv)} \quad (6.100)$$

which is obtained from  $\Phi_{10}$  using 4D-5D lift. Once we go from five to four dimensions there is an additional contribution coming from the KK monopole excitations [26]. This explains the factor  $\eta(\rho)^{24}$  in the formula above.



In the limit of large charges <sup>18</sup> we can rewrite the index as an entropy function [35]

$$C_2(n, Q_1 Q_5, J) \simeq \int \frac{d^2 \tau}{\tau_2^2} e^{-F(\tau_1, \tau_2)} \quad (6.101)$$

with the free energy  $F(\tau_1, \tau_2)$  given by

$$\begin{aligned} F(\tau_1, \tau_2) = & -\frac{\pi}{\tau_2} (n + Q_1 Q_5 |\tau|^2 - \tau_1 J) + 24 \ln \eta(\tau) + 24 \ln \eta(-\bar{\tau}) + 12 \ln 2\tau_2 \\ & + -24 \ln \eta(i/2\tau_2) - 4 \ln \{2 \cosh(\pi \tau_1 / 2\tau_2)\} \\ & - \ln \left[ \frac{1}{4\pi} \left\{ 26 + \frac{2\pi}{\tau_2} (n + Q_1 Q_5 |\tau|^2 - \tau_1 J) + i \frac{24}{\tau_2} \frac{\eta'(i/2\tau_2)}{\eta(i/2\tau_2)} + 4\pi \frac{\tau_1}{\tau_2} \tanh \frac{\pi \tau_1}{2\tau_2} \right\} \right] \end{aligned} \quad (6.102)$$

In the limit of large  $n$  and finite  $Q_1 Q_5$  and  $J$  we use the ansatz that at the saddle point  $\tau_2$  becomes very large. Therefore, neglecting some terms in (6.102), we keep the first square bracket term and approximate  $24 \ln \eta(\tau) + 24 \ln \eta(-\bar{\tau}) \simeq -4\pi\tau_2$  and  $24 \ln \eta(i/2\tau_2) \simeq -4\pi\tau_2$ . The saddle point is at

$$\tau_1^* = \frac{J}{2Q_1 Q_5}, \quad \tau_2^* = \sqrt{\frac{4nQ_1 Q_5 - J^2}{4Q_1 Q_5}} \quad (6.103)$$

which in the limit considered justifies our ansatz. The entropy function evaluated at the saddle point gives the Beckenstein-Hawking entropy of the BMPV black hole

$$\ln C_2 \approx \pi \sqrt{4nQ_1 Q_5 - J^2} \quad (6.104)$$

This limit is also known as Cardy-limit.

Another interesting limit is to consider the case when the number of D1-branes becomes very large while keeping the other charges finite. This case is easier to understand from the type IIA perspective. Using ten dimensional S-duality followed by a T-duality along  $S^1$  we map the D1-D5-P system to a system with  $Q_5$  NS5-branes,  $n$  fundamental strings wrapped along  $\tilde{S}^1$ , the dual circle, and momenta  $Q_1$  along the same circle. The limit  $Q_1$  very large corresponds to the Cardy limit of the low energy theory of the F-NS5 system. This example is interesting due to the presence of NS5-branes. Even though we don't know the microscopic theory, the anomaly coefficients can be computed from the bulk using the technique already described. Moreover since the index is invariant under duality, we can use the answer (6.100) obtained in the type IIB frame and compare it with the macroscopic answer in the IIA frame.

In this case we assume the ansatz that  $\tau_2$  is small at the saddle point. Defining a new variable  $\sigma_1 + i\sigma_2 = \tau^{-1}$  and using modular properties, we rewrite (6.102) in terms of this variable. The limit of small  $\tau_2$  corresponds to large  $\sigma_2$ . The saddle point is at

$$\sigma_1 = -\frac{J}{2(n-1)}, \quad \sigma_2 = \sqrt{\left( Q_1 Q_5 - \frac{J^2}{4(n-1)} \right) / (n+3)} \quad (6.105)$$

---

<sup>18</sup>This limit is particularly different from the limit used in [26] in the sense that only one of the charges is taken to be very large. This is an important difference to consider when performing the asymptotic expansion. We refer the reader to the appendix of [35] where this analysis has been carried out carefully

which in the limit considered justifies our ansatz. The Beckenstein-Hawking entropy becomes

$$\ln C_2 \simeq 2\pi \sqrt{(n+3) \left( Q_1 Q_5 - \frac{J^2}{4(n-1)} \right)} \quad (6.106)$$

In summary,

1. Cardy limit:  $n \rightarrow \infty$ , fixed  $Q_1, Q_5$  charges and angular momenta  $J$ ,

$$\ln B_{2\text{micro}} \simeq \ln B_{2\text{macro}} \simeq 2\pi \sqrt{n Q_1 Q_5 - J^2/4} \quad (6.107)$$

where  $\simeq$  means equality up to corrections suppressed by powers of  $n$ .

2. Anti-Cardy limit:  $Q_1 \rightarrow \infty$ , fixed  $n, Q_5$  charges and angular momenta  $J$ ,

$$\ln B_{2\text{micro}} \simeq \ln B_{2\text{macro}} \simeq 2\pi \sqrt{(n+3) \left( Q_1 Q_5 - \frac{J^2}{4(n-1)} \right)} \quad (6.108)$$

where  $\simeq$  means equality up to corrections suppressed by powers of  $Q_1$ .

### 6.5.2 Macroscopic derivation

The analysis of the Chern-Simons in five dimensions requires a bit more work than in four dimensions. The main difference is the additional  $SU(2)_L$  R-symmetry current dual to the bulk  $SU(2)_L$  gauge field. Because the black hole has a  $S^3$  factor in its near horizon geometry, after reducing down to  $AdS_3$  a  $SU(2)_L$  gauge Chern-Simons will be generated. The Cardy formula which depends explicitly on the anomaly coefficient  $k_L$ , is given by

$$\ln C_{2,\text{macro}} \simeq 2\pi \sqrt{c_L^{\text{macro}}/6 \left( n - \frac{J_1^2}{4k_L^{\text{macro}}} \right)} \quad (6.109)$$

We give the results for both the Cardy and Anti-Cardy limits. For additional details on the computation we refer the reader to [35].

1. Cardy limit:

$$\begin{aligned} c_L^{\text{macro}} &= 6Q_1 Q_5, \quad k_L^{\text{macro}} = Q_1 Q_5 \\ c_R^{\text{macro}} &= 6Q_1 Q_5, \quad k_R^{\text{macro}} = Q_1 Q_5 \end{aligned}$$

$$\ln C_{2,\text{macro}} \approx \pi \sqrt{4n Q_1 Q_5 - J^2}$$

2. Anti-Cardy limit:

$$\begin{aligned} c_L^{\text{macro}} &= 6Q_5(n+3), \quad k_L^{\text{macro}} = Q_5(n-1) \\ c_R^{\text{macro}} &= 6Q_5(n+1), \quad k_R^{\text{macro}} = Q_5(n+1) \end{aligned}$$

$$\ln C_{2,\text{macro}} \approx 2\pi \sqrt{(n+3) \left( Q_1 Q_5 - \frac{J^2}{4(n-1)} \right)}$$

Both the Cardy and Anti-Cardy limits of the macroscopic index are in perfect agreement with the microscopic results.

## 7. Discussion, conclusions and outlook

It is remarkable that a functional integral of string theory in  $AdS_2$  precisely reproduces the first term in the Rademacher expansion that already captures all power-law suppressed corrections to the Bekenstein-Hawking-Wald formula as described in §5.4.3. As we have seen in §5.4.4, the functional integral has all the ingredients to reproduce even the subleading nonperturbative corrections in the Rademacher expansion. It would be interesting to see how string theory functional integral reproduces the detailed number theoretic details of the Kloosterman sum. Since  $d(\Delta)$  is an integer  $W(\Delta)$  would also have to be an integer. This suggests an underlying integral structure in quantum gravity at a deeper level.

Our computation suggests that the bulk  $AdS$  string theory is every bit as fundamental as the boundary  $CFT$ . Even though one sometimes refers to the  $AdS$  computation as macroscopic and thermodynamic, quantum gravity in  $AdS_2$  does not appear to be an emergent, coarse-grained description of the more microscopic boundary theory. Each theory has its own rules of computation. It seems more natural to regard  $AdS/CFT$  holography as an exact strong-weak coupling duality.

So far we have used holography in its original sense to mean a complete accounting of the degrees of freedom associated with the  $AdS_2$  black hole horizon in terms of the states of a  $CFT_1$  in one lower dimension. The  $AdS_2/CFT_1$  correspondence actually extends this idea further to apply correlation functions as well. The boundary  $CFT_1$  has a  $GL(d)$  symmetry that acts upon  $d(q, p)$  zero energy states. The observables of the theory are thus simply  $d \times d$  matrices  $\{M_i\}$ . A precise state-operator correspondence has been suggested [86] that allows one to define, at least formally, the corresponding correlation functions for some of the observables in the bulk theory. In the boundary theory it is easy to define correlation functions of observables as traces of strings of operators such as

$$\text{Tr}(M_1 M_2 \dots M_k) . \quad (7.1)$$

We have seen that localization techniques can be successfully applied for computing the partition function to compute the integer  $d$ . A natural question is if localization can be useful for computing the correlation functions such as above. Such a computation would allow us to recover the discrete information about the microstates of a black hole from observables living in the bulk near the horizon. This of course goes to the heart of the problem of information retrieval from black holes. It is likely that one would need to extend the localization analysis beyond the massless fields to higher string modes to access this information.

The content of the boundary  $CFT_1$  is essentially completely determined by the integer  $d$ . The bulk theory has an elaborate field content and action that depends on the compactification  $K$  and the charges of the black hole. Imagine two different bulk theories  $AdS_2 \times K$  and  $AdS_2 \times K'$  but with the same black hole degeneracy  $d$ . This would suggest that the two string theories near the horizon of two very different black holes in very different compactifications are dual to the same  $CFT_1$ . By transitivity of duality, this would imply that the two string theories themselves are dual to each other. This conclusion seems inescapable from the perspective of the  $CFT_1$ . Note that it is not easy to arrange the situation when the degeneracies of two

different black holes are given by the same integer. For example, if the degeneracy is given by the Fourier coefficients of some modular form, it would be rare, but not impossible, that two such Fourier coefficients are precisely equal.

Our analysis uses an  $\mathcal{N} = 2$  reduction of the full  $\mathcal{N} = 8$  theory by dropping six gravitini multiplets of  $\mathcal{N} = 2$  and the hypermultiplets. This was motivated by the fact that hypermultiplets are flat directions of the classical entropy function and the black hole is not charged under the gauge fields in the gravitini multiplets. We have also ignored D-terms. This is partially justified by the fact that the black hole horizon is supersymmetric and a large class of D-terms are known not to contribute to the Wald entropy as a consequence of this supersymmetry [105]. Our final answer strongly suggests that these assumptions are justified and our reduced theory fully captures the physics. A technical obstacle in justifying these assumptions stems from the fact that the incorporation of the hypermultiplets and the gravitini multiplets would require infinite number of auxiliary fields if all  $\mathcal{N} = 8$  supersymmetries are realized off-shell. It may be possible to make progress in this direction perhaps by using a formulation where only the Q-supersymmetry used for localization is realized off-shell but on all fields of  $\mathcal{N} = 8$  supergravity. Alternatively, it may be possible to repeat the localization analysis in a different off-shell formalism such as the harmonic superspace [141] where all  $\mathcal{N} = 8$  supersymmetries are realized off-shell with infinite number of auxiliary fields; but perhaps only a small number of auxiliary fields get excited for the localizing solution.

We think that the application of localization techniques in  $AdS_3$  could be very interesting as a means to understand the elliptic genus from a gravity perspective. We have already given some exact results in this context. In the last section we saw that in a particular charge regime, namely when only one of the charges is taken to be very large, the asymptotic growth of the index is controlled by the coefficients of the Chern-Simons terms. Even if this seems to follow naturally in a theory which has an holographic description, in this case it is surprising because we had to take into account an exterior contribution for which we don't have an holographic dual.

In establishing an exact  $AdS_2/CFT_1$  holography it is necessary not only to just compute exactly the quantum entropy, for which we give an important contribution, but also to have in hands precise microscopic answers. Duality plays a very important role in this matter. It is therefore important that the results are consistent with the duality symmetries of the theory. Much of this work has been accomplished here for quarter-BPS dyons in  $\mathcal{N} = 4$  string theory. In section §2 we proposed a two dimensional supersymmetric sigma model whose index captures dyons with non-trivial values of  $I$ . Part of the microscopic answer, namely the divisor structure, has already been understood from the bulk perspective in [73]. Inclusion of orbifold geometries is necessary to explain the divisor structure in the microscopic answer (2.53). Although there are still some caveats, mainly concerning the symmetrization of fermion zero modes, the index obtained passes many physical checks and is consistent with duality. The perturbative analysis of a set of two charge configurations presented in §2.5 gives an additional and non-trivial important check. We think this work is worth to be explored in  $\mathcal{N} = 4$  CHL models or in  $\mathcal{N} = 8$  string theory.

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## A. Killing spinors in the attractor geometry

To apply localization arguments, it is necessary to identify the supercharge  $Q$  that squares to the compact bosonic generator  $L - J$ . For this purpose, it is useful to know first the explicit form of the Killing spinors in the on-shell attractor geometry.

Recall that in the superconformal formalism, there are fermionic variations corresponding to  $Q$  as well as  $S$ , which we label by  $\varepsilon$  and  $\eta$  respectively [87]. One can only impose  $Q$ -invariance up to a uniform  $S$ -supertranslation. This corresponds to the fact that the physical supersymmetries in the Poincaré theory are found after the gauge fixing procedure to be a linear combination of these two variations. In general, this combination has a complicated dependence on the other fields as well as the choice of prepotential. The method of [90] is to surpass this problem by finding spinor fields whose variation under  $S$  vanishes. One can then simply use the  $Q$ -invariance conditions for these spinor fields, which by construction is gauge independent. This construction was very useful in [90] to find the half-BPS solution in asymptotically flat space.

However, these gauge-independent supersymmetry transformations then depend on the choice of prepotential and hence the choice of the Lagrangian. This is not well-suited for our purposes since we are really interested in the off-shell localizing solutions that are determined directly by the off-shell supersymmetry transformation without any reference to the prepotential. Moreover, we are only interested in the near horizon geometry which is much simpler to analyze than the full black hole solution including the asymptotic infinity. For the near horizon supersymmetries, we make the simple observation that a choice of the bosonic fields corresponding to the near horizon attractor background leads to a particularly simple choice of gauge for the physical theory, namely  $\eta = 0$ . This choice then permits us to work with the simpler supersymmetry transformations of the superconformal theory.

To see this, we begin by imposing the vanishing of the variations of fermionic fields of the Weyl multiplet:

$$0 = \delta\psi_\mu^i = 2D_\mu\epsilon^i - \frac{1}{8}\gamma_a\gamma_b T^{abij}\gamma_\mu\epsilon_j + \gamma_\mu\eta^i, \quad (\text{A.1})$$

$$0 = \delta\chi^i = -\frac{1}{12}\gamma_a\gamma_b \not{D}T^{abij}\epsilon_j + D\epsilon^i + \frac{1}{12}T_{ab}^{ij}\gamma^a\gamma^b\eta_j, \quad (\text{A.2})$$

$$0 = \delta\phi_\mu^i = -2f_\mu^a\gamma_a\epsilon^i - \frac{1}{4}\not{D}T_{cd}^{ij}\sigma^{cd} + 2D_\mu\eta^i. \quad (\text{A.3})$$

At the attractor values, we have

$$v = \frac{16}{\omega\bar{\omega}}, \quad T_{rt}^- = v\omega, \quad (\text{A.4})$$

and the above variations simplify to

$$\delta\psi_\mu^i = 2D_\mu\epsilon^i - \frac{1}{8}\gamma_a\gamma_b T^{abij}\gamma_\mu\epsilon_j + \gamma_\mu\eta^i, \quad (\text{A.5})$$

$$\delta\chi^i = \frac{1}{12}T_{ab}^{ij}\gamma^a\gamma^b\eta_j, \quad (\text{A.6})$$

$$\delta\phi_\mu^i = 2D_\mu\eta^i. \quad (\text{A.7})$$

From here, we deduce the  $AdS_2 \times S^2$  Killing spinors equations

$$\begin{aligned} D_\mu\epsilon^i &= \frac{1}{16}\gamma_a\gamma_b T^{abij}\gamma_\mu\epsilon_j, \\ D_\mu\epsilon_i &= \frac{1}{16}\gamma_a\gamma_b T^{ab}{}_{ij}\gamma_\mu\epsilon^j \\ \eta_i &= \eta^i = 0. \end{aligned} \quad (\text{A.8})$$

We thus see that  $\eta^i = 0$  as promised. Before solving the equation for  $\epsilon^i, \epsilon_i$ , note that in the Euclidean theory in four dimensions, the spinors should have a symplectic-Majorana condition imposed on them, while in Minkowski spacetime they can be majorana or symplectic-Majorana [88]. In addition, the Weyl projection is not compatible with the majorana condition in the Minkowski case and therefore the left and right-handed spinors are complex conjugate to each other. On the contrary, in the Euclidean case, we can have symplectic Majorana-Weyl spinors but not majorana

$$(\zeta_\pm^i)^* = -i\varepsilon_{ij}(\sigma_1 \times \sigma_2)\zeta_\pm^j, \quad (\text{A.9})$$

where the indices  $i, j$  are  $SU(2)'$  quantum numbers and  $\varepsilon_{ij}$  is the antisymmetric tensor of  $SU(2)$ . In the literature [87] the spinors used obeyed a majorana condition in Minkowski space. They used the convention that positive/negative chirality is correlated with upper/down  $SU(2)'$  indices due to complex conjugation. Since the killing spinor equations A.8 were derived from the Lorentzian theory, we shall use an ansatz which reproduces the killing spinor equations in Euclidean  $AdS_2 \times S^2$ . The ansatz is the following

$$\epsilon_i = i\varepsilon_{ij}\xi_-^j \quad (\text{A.10})$$

$$\epsilon^i = \xi_+^i \quad (\text{A.11})$$

Note that we explicitly show the chirality of the spinor. We should therefore solve the Killing spinor condition for an unconstrained Dirac spinor  $\xi^i = \xi_+^i + \xi_-^i$ , double the space and then impose the above constraint (A.9). We represent the Dirac spinor  $\xi$  as a direct product  $\xi =$

$\xi_{AdS_2} \otimes \xi_{S^2}$  where  $\xi_{AdS_2}$  and  $\xi_{S^2}$  are two component spinors, and use the following gamma matrix representation

$$\gamma_\theta = \sqrt{v} \sinh \eta \sigma_1 \otimes 1, \quad \gamma_\eta = \sqrt{v} \sigma_2 \otimes 1, \quad \gamma_\phi = \sqrt{v} \sin \psi \sigma_3 \otimes \sigma_1, \quad \gamma_\psi = \sqrt{v} \sigma_3 \otimes \sigma_2, \quad (\text{A.12})$$

where  $v \equiv v_1 (= v_2)$  is the classical size of the  $AdS_2$  (and the  $S^2$ ).

Equations (A.8) simplify to the diagonal form

$$D_\mu \xi_{AdS_2}^i = \frac{\omega}{|\omega|} \frac{i}{2} (\sigma_3 \times 1) \gamma_\mu \xi_{AdS_2}^i, \quad (\text{A.13})$$

$$D_j \xi_{S^2}^i = \frac{\omega}{|\omega|} \frac{i}{2} (\sigma_3 \times 1) \gamma_j \xi_{S^2}^i. \quad (\text{A.14})$$

$$(\text{A.15})$$

which are easily solved [142]. In the bispinor basis

$$\begin{aligned} \xi &= a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\quad + a_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} + a_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\equiv \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \end{aligned} \quad (\text{A.16})$$

the solutions are (this is assuming that  $w \in \mathbb{R}^+$ , and we have fixed a certain normalization for the spinors):

$$\begin{aligned} \xi_{--}^i &= 2 e^{-\frac{i}{2}(\theta+\phi)} \begin{pmatrix} \cosh \frac{\eta}{2} \cos \frac{\psi}{2} \\ \sinh \frac{\eta}{2} \cos \frac{\psi}{2} \\ -\cosh \frac{\eta}{2} \sin \frac{\psi}{2} \\ -\sinh \frac{\eta}{2} \sin \frac{\psi}{2} \end{pmatrix}, & \xi_{-+}^i &= 2 e^{-\frac{i}{2}(\theta-\phi)} \begin{pmatrix} \cosh \frac{\eta}{2} \sin \frac{\psi}{2} \\ \sinh \frac{\eta}{2} \sin \frac{\psi}{2} \\ \cosh \frac{\eta}{2} \cos \frac{\psi}{2} \\ \sinh \frac{\eta}{2} \cos \frac{\psi}{2} \end{pmatrix}, \\ \xi_{+-}^i &= 2 e^{\frac{i}{2}(\theta-\phi)} \begin{pmatrix} \sinh \frac{\eta}{2} \cos \frac{\psi}{2} \\ \cosh \frac{\eta}{2} \cos \frac{\psi}{2} \\ -\sinh \frac{\eta}{2} \sin \frac{\psi}{2} \\ -\cosh \frac{\eta}{2} \sin \frac{\psi}{2} \end{pmatrix}, & \xi_{++}^i &= 2 e^{\frac{i}{2}(\theta+\phi)} \begin{pmatrix} \sinh \frac{\eta}{2} \sin \frac{\psi}{2} \\ \cosh \frac{\eta}{2} \sin \frac{\psi}{2} \\ \sinh \frac{\eta}{2} \cos \frac{\psi}{2} \\ \cosh \frac{\eta}{2} \cos \frac{\psi}{2} \end{pmatrix} \end{aligned} \quad (\text{A.17})$$

As explained above, we should impose a symplectic-Majorana condition on the spinors. In the

above basis, equation (A.9) implies:

$$\begin{aligned}\xi_{++}^+ &= (\xi_{--}^-)^* \\ \xi_{-+}^+ &= (\xi_{+-}^-)^* \\ \xi_{++}^- &= (-\xi_{--}^+)^* \\ \xi_{-+}^- &= (-\xi_{+-}^+)^*\end{aligned}$$

where the star is not the ordinary complex conjugation but the complex conjugation condition as defined by the symplectic-majorana condition.

One can now identify the spinors  $\epsilon_{ra}^i$  as the generators of  $G_r^{ia}$ , the supercharges of the near horizon  $\mathcal{N} = 4$  superalgebra §4.1.2. The real combinations  $Q_\mu, \tilde{Q}_\mu, \mu = 1, \dots, 4$  are generated by the combinations:

$$\begin{aligned}\zeta_1 &= \xi_{++}^+ + \xi_{--}^-, & \tilde{\zeta}_1 &= \xi_{-+}^+ + \xi_{+-}^-, \\ \zeta_2 &= -i(\xi_{++}^+ - \xi_{--}^-), & \tilde{\zeta}_2 &= -i(\xi_{-+}^+ - \xi_{+-}^-), \\ \zeta_3 &= -i(\xi_{++}^- + \xi_{--}^+), & \tilde{\zeta}_3 &= -i(\xi_{-+}^- + \xi_{+-}^+), \\ \zeta_4 &= \xi_{++}^- - \xi_{--}^+, & \tilde{\zeta}_4 &= \xi_{-+}^- - \xi_{+-}^+,\end{aligned}\tag{A.18}$$

We can easily see that these killing spinors are real under the complex conjugation condition defined by (A.9). As an instructive exercise take for example  $\zeta^1$ . The  $SU(2)'$  components are  $\zeta^{1+} = \xi_{++}^+$  and  $\zeta^{1-} = \xi_{--}^-$ . Both are complex conjugate to each other

$$\begin{aligned}(\zeta^{1+})^* &= -i\varepsilon_{+-}(\sigma_1 \times \sigma_2)\zeta^{1-} \\ (\zeta^{1-})^* &= -i\varepsilon_{-+}(\sigma_1 \times \sigma_2)\zeta^{1+}\end{aligned}$$

## A.1 Supersymmetry variations

Recall that the supersymmetry variations for fermions and scalars of the vector multiplets in Minkowski theory are [87]

$$\begin{aligned}\delta X^I &= \bar{\epsilon}^i \Omega_i^I \\ \delta \bar{X}^I &= \bar{\epsilon}_i \Omega^{Ii} \\ \delta \Omega_i^I &= 2\partial X^I \epsilon_i + \frac{1}{2}\varepsilon_{ij}\mathcal{F}^{I\mu\nu-}\gamma_\mu\gamma_\nu\epsilon^j + Y_{ij}^I\epsilon^j + 2X^I\eta_i \\ \delta \Omega^{Ii} &= 2\partial \bar{X}^I \epsilon^i + \frac{1}{2}\varepsilon^{ij}\mathcal{F}^{I\mu\nu+}\gamma_\mu\gamma_\nu\epsilon_j + Y^{Iij}\epsilon_j + 2\bar{X}^I\eta^i\end{aligned}$$

where  $\Omega_i$  has positive chirality while  $\Omega^i$  has negative chirality. Changing basis from the  $\epsilon$  spinors to the  $\zeta$  spinors using (A.10), we can reexpress the susy variations as

$$\begin{aligned}\delta X^I &= -(\zeta_+^i)^\dagger \lambda_+^{Ii} \\ \delta \bar{X}^I &= -(\zeta_-^i)^\dagger \lambda_-^{Ii} \\ \delta \lambda_+^{Ii} &= \frac{1}{2}(F_{\mu\nu}^{I-} - \frac{1}{4}\bar{X}^I T_{\mu\nu}^-)\gamma^\mu\gamma^\nu\zeta_+^i + 2i\partial X^I\zeta_-^i + Y_j^{Ii}\zeta_+^j \\ \delta \lambda_-^{Ii} &= \frac{1}{2}(F_{\mu\nu}^{I+} - \frac{1}{4}X^I T_{\mu\nu}^+)\gamma^\mu\gamma^\nu\zeta_-^i + 2i\partial \bar{X}^I\zeta_+^i + Y_j^{Ii}\zeta_-^j\end{aligned}\tag{A.19}$$



where  $\lambda$  are related to  $\Omega$  spinors by

$$\Omega_i = \varepsilon_{ij} \lambda_-^j \quad \Omega^i = -i \lambda_+^i. \quad (\text{A.20})$$

Under a transformation generated by  $\zeta_i$  or  $\tilde{\zeta}_i$ , given in (A.18), we can show that the action of  $\delta^2$  is  $L - J$  or  $L + J$  respectively

$$\delta^2 X^I = -(\zeta_+^i)^\dagger \delta \lambda_+^{Ii} = 2i(\zeta_+^i)^\dagger \not{\partial} X^I \zeta_-^i \quad (\text{A.21})$$

$$\delta^2 \overline{X}^I = -(\zeta_-^i)^\dagger \delta \lambda_-^{Ii} = 2i(\zeta_-^i)^\dagger \not{\partial} \overline{X}^I \zeta_+^i \quad (\text{A.22})$$

where the remaining contractions vanish identically for the spinors chosen. After a straightforward computation we find

$$\delta^2 X^I = -2i(\partial_\theta - \partial_\phi) X^I = 2(L - J) X^I \quad (\text{A.23})$$

$$\delta^2 \overline{X}^I = -2i(\partial_\theta - \partial_\phi) \overline{X}^I = 2(L - J) \overline{X}^I. \quad (\text{A.24})$$

## B. Some aspects of the superconformal multiplet calculus

In this appendix, we shall summarize some aspects of the superconformal multiplet calculus which we briefly presented in §4.2. We shall first present the supersymmetry variation of the various multiplets. We shall then present the invariant Lagrangian density formula for a chiral multiplet. We shall then present the rule which defines the various components of a scalar function of chiral superfields *e.g.* the prepotential superfield  $\mathbf{F}(\mathbf{X}^{\mathbf{I}})$ . These are the basic ingredients that go into building the superconformal action. We shall borrow the presentation of the recent [105] wherein a lot of these facts (and more) have been collected, this can be referred to for more details.

The invariance of the bulk Lagrangian under the superconformal transformations are well established, we provide these details for the sake of completeness. Using the same transformations, in another appendix, we shall sketch the supersymmetry invariance of our conjectured boundary action. This, as far as we know, is new, and there is scope to develop it further.

As in the text,  $\epsilon_i$  and  $\eta_i$  denote the parameters of the  $Q$  and  $S$  supersymmetry transformations. The transformation rules for a chiral multiplet of Weyl weight  $w$  are:

$$\begin{aligned} \delta A &= \bar{\epsilon}^i \Psi_i, \\ \delta \Psi_i &= 2 \not{D} A \epsilon_i + B_{ij} \epsilon^j + \frac{1}{2} \gamma^{ab} F_{ab}^- \varepsilon_{ij} \epsilon^j + 2 w A \eta_i, \\ \delta B_{ij} &= 2 \bar{\epsilon}_{(i} \not{D} \Psi_{j)} - 2 \bar{\epsilon}^k \Lambda_{(i} \varepsilon_{j)k} + 2(1 - w) \bar{\eta}_{(i} \Psi_{j)}, \\ \delta F_{ab}^- &= \frac{1}{2} \varepsilon^{ij} \bar{\epsilon}_i \not{D} \gamma_{ab} \Psi_j + \frac{1}{2} \bar{\epsilon}^i \gamma_{ab} \Lambda_i - \frac{1}{2} (1 + w) \varepsilon^{ij} \bar{\eta}_i \gamma_{ab} \Psi_j, \\ \delta \Lambda_i &= -\frac{1}{2} \gamma^{ab} \not{D} F_{ab}^- \epsilon_i - \not{D} B_{ij} \varepsilon^{jk} \epsilon_k + C \varepsilon_{ij} \epsilon^j + \frac{1}{4} (\not{D} A \gamma^{ab} T_{abij} + w A \not{D} \gamma^{ab} T_{abij}) \varepsilon^{jk} \epsilon_k \\ &\quad - 3 \gamma_a \varepsilon^{jk} \epsilon_k \bar{\chi}_{[i} \gamma^a \Psi_{j]} - (1 + w) B_{ij} \varepsilon^{jk} \eta_k + \frac{1}{2} (1 - w) \gamma^{ab} F_{ab}^- \eta_i, \\ \delta C &= -2 \varepsilon^{ij} \bar{\epsilon}_i \not{D} \Lambda_j - 6 \bar{\epsilon}_i \chi_j \varepsilon^{ik} \varepsilon^{jl} B_{kl} \\ &\quad - \frac{1}{4} \varepsilon^{ij} \varepsilon^{kl} ((w - 1) \bar{\epsilon}_i \gamma^{ab} \not{D} T_{abjk} \Psi_l + \bar{\epsilon}_i \gamma^{ab} T_{abjk} \not{D} \Psi_l) + 2 w \varepsilon^{ij} \bar{\eta}_i \Lambda_j. \end{aligned} \quad (\text{B.1})$$

The independent fields of the Weyl multiplet transform as follows,

$$\begin{aligned}
\delta e_\mu^a &= \bar{\epsilon}^i \gamma^a \psi_{\mu i} + \bar{\epsilon}_i \gamma^a \psi_\mu^i, \\
\delta \psi_\mu^i &= 2 \mathcal{D}_\mu \epsilon^i - \frac{1}{8} T_{ab}^{ij} \gamma^{ab} \gamma_\mu \epsilon_j - \gamma_\mu \eta^i \\
\delta b_\mu &= \frac{1}{2} \bar{\epsilon}^i \phi_{\mu i} - \frac{3}{4} \bar{\epsilon}^i \gamma_\mu \chi_i - \frac{1}{2} \bar{\eta}^i \psi_{\mu i} + \text{h.c.} + \Lambda_K^a e_{\mu a}, \\
\delta A_\mu &= \frac{1}{2} i \bar{\epsilon}^i \phi_{\mu i} + \frac{3}{4} i \bar{\epsilon}^i \gamma_\mu \chi_i + \frac{1}{2} i \bar{\eta}^i \psi_{\mu i} + \text{h.c.}, \\
\delta \mathcal{V}_\mu^i{}_j &= 2 \bar{\epsilon}_j \phi_\mu^i - 3 \bar{\epsilon}_j \gamma_\mu \chi^i + 2 \bar{\eta}_j \psi_\mu^i - (\text{h.c.}; \text{traceless}), \\
\delta T_{ab}^{ij} &= 8 \bar{\epsilon}^{[i} R(Q)_{ab}{}^{j]}, \\
\delta \chi^i &= -\frac{1}{12} \gamma^{ab} \not{D} T_{ab}^{ij} \epsilon_j + \frac{1}{6} R(\mathcal{V})_{\mu\nu}{}^i{}_j \gamma^{\mu\nu} \epsilon^j - \frac{1}{3} i R_{\mu\nu}(A) \gamma^{\mu\nu} \epsilon^i + D \epsilon^i + \frac{1}{12} \gamma_{ab} T^{abij} \eta_j, \\
\delta D &= \bar{\epsilon}^i \not{D} \chi_i + \bar{\epsilon}_i \not{D} \chi^i,
\end{aligned} \tag{B.2}$$

where

$$R(Q)_{\mu\nu}{}^i{}_j = 2 \mathcal{D}_{[\mu} \psi_{\nu]}^i - \gamma_{[\mu} \phi_{\nu]}^i - \frac{1}{8} T^{abij} \gamma_{ab} \gamma_{[\mu} \psi_{\nu]}^j. \tag{B.3}$$

Based on these two multiplets, one can write down a Lagrangian density for the chiral multiplet which is invariant under the superconformal transformations:

$$\begin{aligned}
e^{-1} \mathcal{L} &= C - \varepsilon^{ij} \bar{\psi}_{\mu i} \gamma^\mu \Lambda_j - \frac{1}{8} \bar{\psi}_{\mu i} T_{abjk} \gamma^{ab} \gamma^\mu \Psi_l \varepsilon^{ij} \varepsilon^{kl} - \frac{1}{16} A (T_{abij} \varepsilon^{ij})^2 \\
&\quad - \frac{1}{2} \bar{\psi}_{\mu i} \gamma^{\mu\nu} \psi_{\nu j} B_{kl} \varepsilon^{ik} \varepsilon^{jl} + \varepsilon^{ij} \bar{\psi}_{\mu i} \psi_{\nu j} (F^{-\mu\nu} - \frac{1}{2} A T^{\mu\nu}{}_{kl} \varepsilon^{kl}) \\
&\quad - \frac{1}{2} \varepsilon^{ij} \varepsilon^{kl} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_{\mu i} \psi_{\nu j} (\bar{\psi}_{\rho k} \gamma_\sigma \Psi_l + \bar{\psi}_{\rho k} \psi_{\sigma j} A).
\end{aligned} \tag{B.4}$$

This density is built such that the variation of the Lagrangian is equal to a total derivative in spacetime.

The Lagrangian for vector multiplets is based on first viewing the gauge invariant quantities of the vector multiplet as a reduced chiral multiplet with weight  $w = 1$ . The components are:

$$\begin{aligned}
A|_{\text{vector}} &= X, \\
\Psi_i|_{\text{vector}} &= \Omega_i, \\
B_{ij}|_{\text{vector}} &= Y_{ij} = \varepsilon_{ik} \varepsilon_{jl} Y^{kl}, \\
F_{ab}^-|_{\text{vector}} &= (\delta_{ab}{}^{cd} - \frac{1}{2} \varepsilon_{ab}{}^{cd}) e_c{}^\mu e_d{}^\nu \partial_{[\mu} A_{\nu]} \\
&\quad + \frac{1}{4} [\bar{\psi}_\rho{}^i \gamma_{ab} \gamma^\rho \Omega^j + \bar{X} \bar{\psi}_\rho{}^i \gamma^{\rho\sigma} \gamma_{ab} \psi_{\sigma}{}^j - \bar{X} T_{ab}{}^{ij}] \varepsilon_{ij}, \\
\Lambda_i|_{\text{vector}} &= -\varepsilon_{ij} \not{D} \Omega^j, \\
C|_{\text{vector}} &= -2 \square_c \bar{X} - \frac{1}{4} F_{ab}^+ T^{ab}{}_{ij} \varepsilon^{ij} - 3 \bar{\chi}_i \Omega^i.
\end{aligned} \tag{B.5}$$

The transformations of the vector multiplet are:

$$\begin{aligned}
\delta X &= \bar{\epsilon}^i \Omega_i, \\
\delta \Omega_i &= 2 \not{D} X \epsilon_i + \frac{1}{2} \varepsilon_{ij} F_{\mu\nu} \gamma^{\mu\nu} \epsilon^j + Y_{ij} \epsilon^j + 2 X \eta_i, \\
\delta A_\mu &= \varepsilon^{ij} \bar{\epsilon}_i (\gamma_\mu \Omega_j + 2 \psi_{\mu j} X) + \varepsilon_{ij} \bar{\epsilon}^i (\gamma_\mu \Omega^j + 2 \psi_\mu{}^j \bar{X}), \\
\delta Y_{ij} &= 2 \bar{\epsilon}_{(i} \not{D} \Omega_{j)} + 2 \varepsilon_{ik} \varepsilon_{jl} \bar{\epsilon}^{(k} \not{D} \Omega^{l)}.
\end{aligned} \tag{B.6}$$

One then has to choose a meromorphic homogeneous function  $F$  of weight 2 and build the multiplet  $\mathbf{F}(\mathbf{X}^I)$  with lowest component  $F(X^I)$ . The components of this is given in terms of the components of the vector multiplet as follows:

$$\begin{aligned}
A|_F &= F(A) , \\
\Psi_i|_F &= F(A)_I \Psi_i^I , \\
B_{ij}|_F &= F(A)_I B_{ij}^I - \frac{1}{2} F(A)_{IJ} \bar{\Psi}_{(i}^I \Psi_{j)}^J , \\
F_{ab}^-|_F &= F(A)_I F_{ab}^{-I} - \frac{1}{8} F(A)_{IJ} \varepsilon^{ij} \bar{\Psi}_i^I \gamma_{ab} \Psi_j^J , \\
\Lambda_i|_F &= F(A)_I \Lambda_i^I - \frac{1}{2} F(A)_{IJ} \left[ B_{ij}^I \varepsilon^{jk} \Psi_k^J + \frac{1}{2} F_{ab}^{-I} \gamma^{ab} \Psi_k^J \right] \\
&\quad + \frac{1}{48} F(A)_{IJK} \gamma^{ab} \Psi_i^I \varepsilon^{jk} \bar{\Psi}_j^J \gamma_{ab} \Psi_k^K , \\
C|_F &= F(A)_I C^I - \frac{1}{4} F(A)_{IJ} \left[ B_{ij}^I B_{kl}^J \varepsilon^{ik} \varepsilon^{jl} - 2 F_{ab}^{-I} F^{-abJ} + 4 \varepsilon^{ik} \bar{\Lambda}_i^I \Psi_j^J \right] , \\
&\quad + \frac{1}{4} F(A)_{IJK} \left[ \varepsilon^{ik} \varepsilon^{jl} B_{ij}^I \Psi_k^J \Psi_l^K - \frac{1}{2} \varepsilon^{kl} \bar{\Psi}_k^I F_{ab}^{-J} \gamma^{ab} \Psi_l^K \right] \\
&\quad + \frac{1}{192} F(A)_{IJKL} \varepsilon^{ij} \bar{\Psi}_i^I \gamma_{ab} \Psi_j^J \varepsilon^{kl} \bar{\Psi}_k^K \gamma_{ab} \Psi_l^L .
\end{aligned} \tag{B.7}$$

## C. Boundary terms and supersymmetry of the renormalized action

In §4.3.3, we conjectured the boundary action (4.77)

$$\mathcal{S}_{\text{bdry}} = -2\pi r_0 \left( \frac{q_I e_*^I}{2} + i (F(X_*^I) - \bar{F}(X_*^I)) \right) . \tag{C.1}$$

so that  $\mathcal{S}_{\text{ren}}$  is finite. We also mentioned that this action is supersymmetric. In this appendix, we shall discuss the action  $\mathcal{S}_{\text{ren}}$ , and show that it is supersymmetric.

To motivate this, we note that we can rewrite  $\mathcal{S}_{\text{ren}}$  as the sum of two pieces

$$\begin{aligned}
\mathcal{S}_{\text{ren}} &= \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{bdry}} + i \frac{q}{2} \oint A \\
&= (\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{bdry}}^1) + \left( \frac{i}{2} q_I \int_0^{2\pi} A_\theta^I d\theta + \mathcal{S}_{\text{bdry}}^2 \right) ,
\end{aligned} \tag{C.2}$$

where we have split the boundary action (4.77) into a sum of two pieces:

$$\mathcal{S}_{\text{bdry}} = \mathcal{S}_{\text{bdry}}^1 + \mathcal{S}_{\text{bdry}}^2 , \tag{C.3}$$

$$\mathcal{S}_{\text{bdry}}^1 = - \int_0^{2\pi} i \left[ F(X) - \bar{F}(X) \right]_{\text{bdry}} e_\theta^\theta d\theta , \tag{C.4}$$

$$\mathcal{S}_{\text{bdry}}^2 = - \int_0^{2\pi} \frac{q_I}{2} \left[ X^I + \bar{X}^I \right]_{\text{bdry}} e_\theta^\theta d\theta . \tag{C.5}$$

Here,  $e^\theta = \sinh \eta_0$  is the induced vielbein on the boundary. To verify (C.3), we use the same algebra used in (4.74), namely, an expansion of the field  $X^I$  into its fixed part  $X_*^I$  and varying part which is  $\mathcal{O}(1/r_0)$ , followed by a Taylor expansion and the use of attractor equations.

With such a split of the action, the two pieces in (C.2) have a very natural interpretation as we discuss below. We will show further that each of them is finite and supersymmetric, implying the same for  $\mathcal{S}_{\text{ren}}$ .

Recall that the bulk action (4.73) evaluated on the solution can be written as the difference of two pieces

$$\mathcal{S}_{\text{bulk}} = 2\pi i r_0 \left[ F(X^I) - \overline{F(X^I)} \right]_{\text{bdry}} - 2\pi i \left[ F(X^I) - \overline{F(X^I)} \right]_{\text{origin}} . \quad (\text{C.6})$$

We see that  $\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{bdry}}^1$  is manifestly finite. Thus,  $\mathcal{S}_{\text{bdry}}^1$  has the natural interpretation of a canonical boundary term which cancels the boundary part of the bulk action, so that any variation of  $\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{bdry}}^1$  will be finite and not contain boundary terms.

The second piece of the boundary action combines with the Wilson line to give the operator

$$\exp \left[ -\frac{i}{2} q_I \int_0^{2\pi} A_\theta^I d\theta - \mathcal{S}_{\text{bdry}}^2 \right] = \exp \left[ -\frac{i}{2} q_I \int_0^{2\pi} \left( A_\theta^I + i e_\theta^\theta (X^I + \overline{X}^I) \right) d\theta \right] \quad (\text{C.7})$$

This operator has the natural interpretation as the supersymmetric Wilson line of gauge theory [100, 101]. Recalling the boundary behavior of the fields

$$-\frac{i}{2} q_I \int_0^{2\pi} A_\theta^I d\theta = -\pi q_I e_*^I r_0 (1 + \mathcal{O}(1/r_0)) , \quad (\text{C.8})$$

$$-\frac{i}{2} q_I \int_0^{2\pi} i e_\theta^\theta (X^I + \overline{X}^I) d\theta = \pi q_I r_0 \left( X_*^I + \overline{X}_*^I + \mathcal{O}(1/r_0) \right) , \quad (\text{C.9})$$

$$= \pi q_I e_*^I r_0 (1 + \mathcal{O}(1/r_0)) , \quad (\text{C.10})$$

it is easy to see that this operator is manifestly finite.

Evaluated on the solutions  $A_\theta^I = -i e_*^I (r_0 - 1)$ ,  $X^I = X_*^I + \frac{C^I}{r_0}$ ,  $\overline{X}^I = \overline{X}_*^I + \frac{C^I}{r_0}$  that we consider in §4.3, we see that the two pieces of the renormalized action (C.2) above give the two pieces of the final renormalized action (4.78) which we found in §4.3.3, as indeed should happen.

In the rest of the appendix, we shall sketch the proof of supersymmetry of these two operators. The supersymmetry of the operator (C.7) above follows from the transformation rules of  $X^I$  and  $A_\mu^I$  of the vector multiplet (B.6). We use the fact that the Killing spinors obey

$$\zeta^i = \varepsilon_{ij} \gamma^0 \zeta^j . \quad (\text{C.11})$$

The extra term in the variation of the vector field which is proportional to the gravitino is cancelled by the variation of the vielbein in the definition of the super Wilson line. This is the new ingredient in the super Wilson line of a gravitational theory compared to that of gauge theory.

Now we come to the supersymmetry of the combination  $\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{bdry}}^1$ . The statement that  $\mathcal{S}_{\text{bulk}}$  is supersymmetric [13, 15, 14] really means that the variation of  $\mathcal{S}_{\text{bulk}}$  is a boundary term which can be ignored in certain circumstances. In our situation, there is a non-trivial boundary,

and therefore what we need to check is that the variation of the bulk Lagrangian is indeed equal to the derivative of the boundary Lagrangian.

To investigate this, we need to understand the structure of the Lagrangian built using the chiral superfield [16]. In the case of rigid supersymmetry, the variation of the top component of the chiral superfield is a total derivative in spacetime, and therefore the top component (picked by a chiral superspace integral) is an invariant Lagrangian. For chiral superfields coupled to superconformal gravity, the transformation rules undergo a modification and the derivatives become covariant derivatives, and there are additional terms in the variation (B.1) of the top component  $C$ . The invariant Lagrangian density (B.4) contains new terms whose variation cancel the additional non-derivative terms present in  $\delta C$ .

The net result of this procedure is that the variation of the invariant Lagrangian is equal to the total derivative terms that are present in the variation of the top component  $C$  of the chiral multiplet, *i.e.* essentially one can drop the extra terms which arise due to the covariantization. As an example, the term proportional to the auxiliary field  $B_{ij}$  in  $\delta C$  contains  $\chi_i$  which is an auxiliary field of the superconformal multiplet constrained to be proportional to  $R(Q)^i$ . This term is cancelled by the term proportional to  $B_{ij}$  in the higher corrections to the Lagrangian density (B.4) after solving for the auxiliary field  $\chi$  in terms of the gravitini.

Looking at the  $Q$  variation (B.1) of a chiral multiplet of weight  $w = 2$ , we see that the variation of  $C$  contains two total derivative pieces

$$- 2\varepsilon^{ij}\partial(\bar{\epsilon}_i \Lambda_j) , \quad (\text{C.12})$$

and

$$- \frac{1}{4}\varepsilon^{ij}\varepsilon^{k\ell}((\partial\bar{\epsilon}_i \gamma^{ab} T_{abjk})\Psi_\ell + \gamma^{ab} T_{abjk} \partial(\bar{\epsilon}_i \Psi_\ell)) = -\frac{1}{4}\varepsilon^{ij}\varepsilon^{k\ell} \partial(\bar{\epsilon}_i \gamma^{ab} T_{abjk} \Psi_\ell) . \quad (\text{C.13})$$

In our problem where we have a bunch of vector multiplets, the way to build a Lagrangian is by using the homogeneous function  $F(X^I)$ . One first builds a chiral multiplet  $\mathbf{F}(\mathbf{X}^{\mathbf{I}})$  whose bottom component is  $F(X^I)$ , and then uses the invariant Lagrangian described above for this chiral multiplet. The variation of our Lagrangian is therefore equal to the total derivative terms that appear in the variation of the top component of the chiral multiplet  $\mathbf{F}(\mathbf{X}^{\mathbf{I}})$ . Looking at the form of the components of this superfield (B.7), and then substituting for the components of the reduced chiral multiplet (B.5), we find that the first type of total derivative term from integration of (C.12) is

$$\begin{aligned} & - 2\varepsilon^{ji} \int_{\text{bulk}} \partial(\bar{\epsilon}_j \Lambda_i)|_F \\ & = -2\varepsilon^{ji} \int_{\text{bulk}} \partial \left( -\bar{\epsilon}_j F(X)_I \varepsilon_{ik} \not{D} \Omega^{kI} - \frac{1}{2} \bar{\epsilon}_j F(X)_{IJ} [B_{ij}^I \varepsilon^{jk} \Omega_k^J + \frac{1}{2} \bar{\epsilon}_j F_{ab}^{-I} \gamma^{ab} \Omega_k^J] \right. \\ & \quad \left. + \frac{1}{48} \bar{\epsilon}_j F(X)_{IJK} \gamma^{ab} \Omega_i^I \varepsilon^{k\ell} \bar{\Omega}_k^J \gamma_{ab} \Omega_\ell^K \right) . \end{aligned} \quad (\text{C.14})$$

We are interested in the bosonic part of the boundary counterterm Lagrangian. The third term on the right hand side contains three fermions and so cannot appear from the variation of a

pure bosonic term, so we shall ignore that term here. The second term on the RHS proportional to  $F_{IJ}$  is equal to the variation of  $F_{IJ}\Omega^I\Omega^J$  minus a total derivative term on the boundary. We can therefore drop this term since it is fermionic. Using the variation  $\delta X^I = \bar{\epsilon}^i\Omega_i^I$ , the first term on the RHS is proportional to the variation of the derivative of  $F_I$ , which integrates to zero on the closed boundary, and therefore does not produce any boundary counterterms.

This leaves us with the second term (C.13) which gives rise to a boundary term

$$-\frac{1}{4}\varepsilon^{ij}\varepsilon^{k\ell}\int_{\text{bulk}}\not{\partial}\left(\bar{\epsilon}_i\gamma^{ab}T_{abjk}\Psi_\ell|_F\right)=-\frac{1}{4}\varepsilon^{ij}\varepsilon^{k\ell}\int_{\text{bulk}}\not{\partial}\left(\bar{\epsilon}_i\gamma^{ab}T_{abjk}F(X)_I\Omega_\ell^I\right). \quad (\text{C.15})$$

Now, the variation of  $T_{abjk}$  (B.2) is proportional to the curvature  $R(Q)_{ab}$  which integrates to zero on the boundary. Therefore,  $T_{abjk}$  can be treated as a constant on the boundary for the purpose of supersymmetry variations. Plugging in the attractor value for  $T_{abjk}$ , and using  $\delta X^I = \bar{\epsilon}^i\Omega_i^I$  again, and the Killing spinor relation (C.11), we see that the remaining term (C.15) is equal and opposite to the variation of the boundary term  $\mathcal{S}_{\text{bdry}}^1$ , thus showing that the supersymmetry variation of  $\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{bdry}}^1$  vanishes.

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